Problem 1. Given a number $a \in [0,1)$ with binary representation $[b_1 b_2 \dots b_n]$

$$a = b_1 2^{-1} + b_2 2^{-2} + \cdots + b_n 2^{-n}$$

If b has a binary representation with $[b_1 b_2 \dots b_n]$ as prefix, then

$$b = b_1 2^{-1} + b_2 2^{-2} + \cdots + b_n 2^{-n} + b_{n+1} 2^{-(n+1)} + \cdots$$

Therefore,

$$b - a = b_{n+1} 2^{-(n+1)} + \cdots$$

Obviously $b - a \geq 0$ and $b \geq a$. To show $b < a + \frac{1}{2^n}$ we note that

$$\begin{aligned} b - a &= b_{n+1} 2^{-(n+1)} + b_{n+2} 2^{-(n+2)} + \cdots \\ &\leq 2^{-(n+1)} + 2^{-(n+2)} + \cdots \\ &< \frac{1}{2^n} \end{aligned}$$

Problem 5.

Letter	Probability	cdf
a_1	.2	$F_X(1) = 0.2$
a_2	.3	$F_X(2) = 0.5$
a_3	.5	$F_X(3) = 1.0$

$l^{(0)} = 0$, $l^{(1)} = 1$.

First letter is a_1:

$$\begin{aligned} l^{(1)} &= 0 + (1-0) \times 0 = 0 \\ u^{(1)} &= 0 + (1-0) \times .2 = .2 \end{aligned}$$

Second letter is a_1:

$$\begin{aligned} l^{(2)} &= 0 + (.2-0) \times 0 = 0 \\ u^{(2)} &= 0 + (.2-0) \times .2 = .04 \end{aligned}$$

Third letter is a_3:

$$\begin{aligned} l^{(3)} &= 0 + (.04-0) \times 0.5 = 0.02 \\ u^{(3)} &= 0 + (.04-0) \times 1.0 = 0.04 \end{aligned}$$

Fourth letter is a_2:

$$\begin{aligned} l^{(4)} &= 0.02 + (.04-0.02) \times 0.2 = 0.024 \\ u^{(4)} &= 0.02 + (.04-0.02) \times 0.5 = 0.03 \end{aligned}$$

Fifth letter is a_3:

$$\begin{aligned} l^{(5)} &= 0.024 + (.03-0.024) \times 0.5 = 0.027 \\ u^{(5)} &= 0.024 + (.03-0.024) \times 1.0 = 0.03 \end{aligned}$$

Sixth letter is a_1:

$$\begin{aligned} l^{(6)} &= 0.027 + (.03 - 0.027) \times 0.0 = 0.027 \\ u^{(6)} &= 0.027 + (.03 - 0.027) \times 0.2 = 0.0276 \end{aligned}$$

Therefore, a possible tag value is 0.0273.

Problem 6.

The tag decodes to the following sequence: $a_3\ a_2\ a_2\ a_1\ a_2\ a_1\ a_3\ a_2\ a_2\ a_3$.

Problem 7.

Letter	Count	Cum_Count
		Cum_Count[0] = 0
a(=1)	37	Cum_Count[1] = 37
b(=2)	38	Cum_Count[2] = 75
c(=3)	25	Cum_Count[3] = 100

(a) Total count = 100,
$m = \lceil \log_2(Total\_Count) \rceil + 2 = 7 + 2 = 9$ bits.

(b) 001010110$\underline{00111100}$. Note that the underlined bit patterns (i.e., EOS) is the lower limit of the final interval, $l^{(7)}$, you can transmit any value between $l^{(7)}$ and $u^{(7)}$.

(c) Decoding of 001010110+EOS should give you *abacabb*.