Partial Differential Equations and Boundary Value Problems†

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† Chapter 12.1 ~ 12.5 in the textbook.

Partial Differential Equation

- A partial differential equation (PDE) is a differential equation that contains partial derivatives of a dependent variable that is a function of at least two independent variables.

- Example: one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

  - $u(x, t)$ is the temperature function of x (position) and t (time) of a heated rod, k is a constant parameter determined by the material of the rod

Linear Partial Differential Equations

- If u is a function of two independent variables x and y, the general form of a linear 2nd-order PDE is given by:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G,$$

  where A, B, C, D, …, G are functions of x and y.

- Example: one-dimensional heat propagation equation can be described by:

$$k\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0.$$

Solving PDE for Separable Functions

- General solutions for PDE are difficult to find, so in practice, we only look for particular solutions.

- In addition to using initial or boundary conditions to constrain our solutions, we often assume that the solution function is separable, that is:

$$u(x, y) = X(x)Y(y).$$

  Thus, we have:

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY', \quad \frac{\partial^2 u}{\partial x^2} = X''Y, \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = XY''.$$

Example: Solving $\dfrac{\partial^2 u}{\partial x^2} = 4\dfrac{\partial u}{\partial y}$. (1/4)

- Let $u(x, y) = X(x)Y(y)$, we have $X''Y = 4XY'$, or

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda,$$

where λ is a constant because changing X won't change Y'/Y and changing Y won't change $X''/4X$. Thus, X and Y must be solutions of

$$X'' + 4\lambda X = 0 \text{ and } Y' + \lambda Y = 0.$$

These are the eigenvalue problem of ODE's†. Consider the three cases: $\lambda = 0$, $\lambda = \alpha^2$, and $\lambda = -\alpha^2$.

† See Section 5.2, example 2.

Example: Solving $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$. (2/4)

- Case I: $\lambda = 0$.

  The two equations become $X'' = 0$ and $Y' = 0$. The general solutions are $X(x) = c_1 + c_2 x$ and $Y(y) = c_3$, respectively.

  Thus, a particular solution of the PDE is $u(x, y) = X(x)Y(y) = (c_1 + c_2 x)c_3 = C_1 + C_2 x$.

Example: Solving $\dfrac{\partial^2 u}{\partial x^2} = 4\dfrac{\partial u}{\partial y}$. (3/4)

- Case II: $\lambda = -\alpha^2$, $\alpha > 0$.

  The two equations becomes $X'' - 4\alpha^2 X = 0$ and $Y' - \alpha^2 Y = 0$. The general solutions becomes $X(x) = c_1 \cosh 2\alpha x + c_2 \sinh 2\alpha x$ and $Y(y) = c_3 e^{\alpha^2 y}$, respectively.

  Thus, a particular solution of the PDE is

  $$\begin{aligned} u(x, y) &= X(x)Y(y) = (c_1 \cosh 2\alpha x + c_2 \sinh 2\alpha x)c_3 e^{\alpha^2 y} \\ &= C_1\, e^{\alpha^2 y} \cosh 2\alpha x + C_2\, e^{\alpha^2 y} \sinh 2\alpha x. \end{aligned}$$

Example: Solving $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$. (4/4)

- Case III: $\lambda = \alpha^2$, $\alpha > 0$.

  The two equations becomes $X'' + 4\alpha^2 X = 0$ and $Y' + \alpha^2 Y = 0$. The general solutions becomes $X(x) = c_1 \cos 2\alpha x + c_2 \sin 2\alpha x$ and $Y(y) = c_3 e^{-\alpha^2 y}$, respectively.

  Thus, a particular solution of the PDE is

$$\begin{aligned} u(x, y) &= X(x)Y(y) = (c_1 \cos 2\alpha x + c_2 \sin 2\alpha x)c_3 e^{-\alpha^2 y} \\ &= C_1\, e^{-\alpha^2 y} \cos 2\alpha x + C_2\, e^{-\alpha^2 y} \sin 2\alpha x. \end{aligned}$$

Superposition Principle for PDE

- If $u_1, u_2, \ldots, u_k$ are solutions of a homogeneous linear partial differential equation, then the linear combination

$$\sum_{i=1}^{k} c_i u_i$$

where the c_i, $i = 1, 2, \ldots, k$, are constants, is also a solution.

The property is true even when $k = \infty$.

Classification of PDE

- The linear 2nd-order partial differential equation with two independent variables,

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G,$$

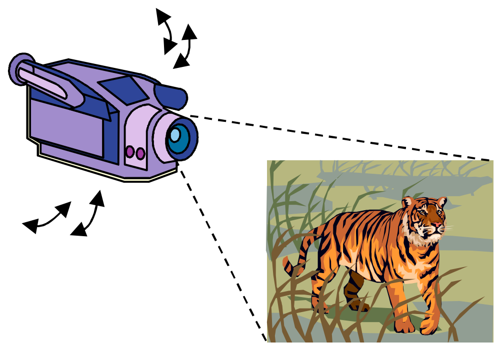
  where A, B, C, D, …, G are real constants, is said to be:
  - Hyperbolic if $B^2 - 4AC > 0$,
  - Parabolic if $B^2 - 4AC = 0$,
  - Elliptic if $B^2 - 4AC < 0$.

Derivation of Classical PDEs

- The derivation of the mathematical model that can be used to explain or predict the behavior of a physical phenomenon is the key to most engineering problems

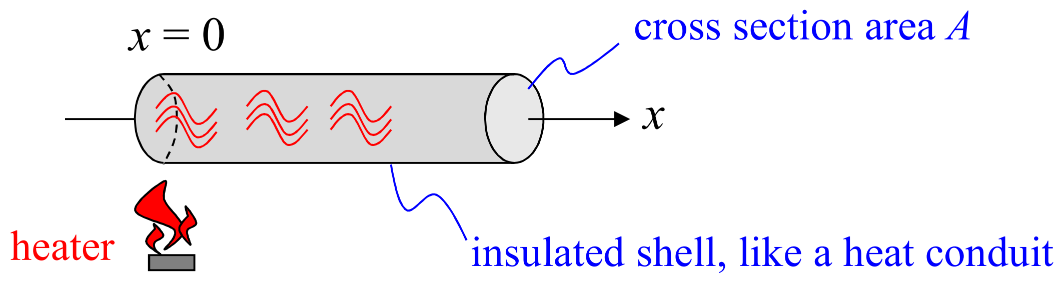
- Example: the optical flow model.
  The motion (dx/dt, dy/dt) of the image pixels $E(x, y, t)$ taken by a camera can be approximated by:

$$\frac{\partial E}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial E}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial E}{\partial t}=0,$$



Derivation of the Heat Equation (1/3)

- Assume that we have a heated rod:



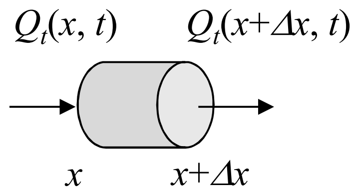
$u(x, t)$ is the temperature of the rod at x and time t.

- From empirical study of thermodynamics:
  - The amount of heat in a element of mass m and temperature u is $Q = \gamma mu$, γ is a constant parameter of the rod.
  - The heat flow $Q_t = -KAu_x$ is the flow of heat in the direction of decreasing temperature, K is a constant parameter of the rod.

† One calorie is the amount of heat required at a pressure of one atmosphere to raise the temperature of one gram of water by one degree Celsius

Derivation of the Heat Equation (2/3)

- The heat content in a segment of the rod is:



$$Q = \gamma m u = \gamma (\rho A \Delta x) u,$$

and the heat flow in this segment is

$$dQ/dt = \gamma \rho A \Delta x \, u_t, \text{ when } \Delta x \to 0 \qquad (1)$$

- Another way to estimate the heat flow is to compute the difference of amount of heat entering/leaving the segment as $\Delta x \to 0$:

$$Q_t(x+\Delta x, t) - Q_t(x, t) = KA[u_x(x+\Delta x, t) - u_x(x, t)] \qquad (2)$$

Derivation of the Heat Equation (3/3)

- Eq (1) and (2) should equal each other as $\Delta x \to 0$, thus

  $KA[u_x(x+\Delta x, t) - u_x(x, t)] \to \gamma\rho A\Delta x\, u_t$, as $\Delta x \to 0$.

  Therefore

  $$\frac{K}{\gamma\rho} \cdot \lim_{\Delta x \to 0} \frac{[u_x(x+\Delta x, t) - u_x(x, t)]}{\Delta x} = u_t$$

  Finally, we obtain the following heat equation:

  $$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$$

  where $k = K/\gamma\rho$ is the thermal diffusivity of the rod.

BVP of the Heat Equation (1/3)

- The solution of a PDE involves arbitrary functions of some dependent variables. For example, the partial DE

$$\frac{\partial u(x,t)}{\partial t} = 0$$

has a general solution $u(x, t) = g(x)$, where $g(x)$ can be any function of x.

Hence, the "initial condition" of a partial DE is a boundary function. In the case of the heated rod, we may have the boundary function $u(x, 0) = f(x)$, where $f(x)$ is the heat function (of x) at time 0.

BVP of the Heat Equation (2/3)

- We may also constrain the temperature function at two ends of the rod and try to solve the PDE. For example,

  $u(0, t) = u(L, t) = 0$, for all $t > 0$.

  A boundary value problem of the heated rod PDE may be as follows:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L,\ t > 0);$$

$$u(0,t) = u(L,t) = 0 \qquad (t > 0),$$

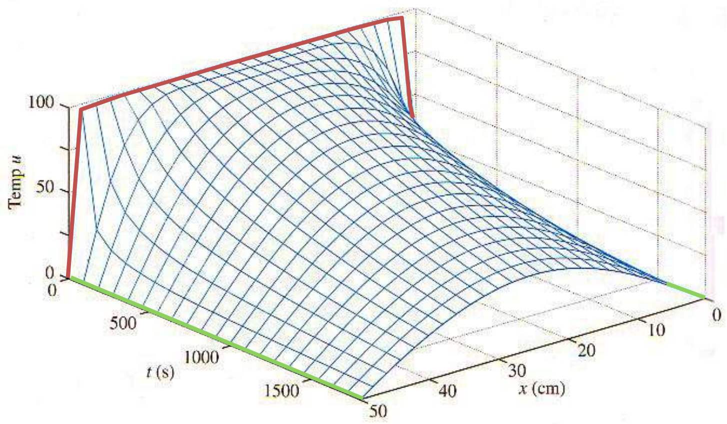
$$u(x,0) = f(x) \qquad (0 < x < L).$$

BVP of the Heat Equation (3/3)

- Another possible boundary condition for the heated rod is that no heat will flow through either end (i.e. both ends are heat-insulated):

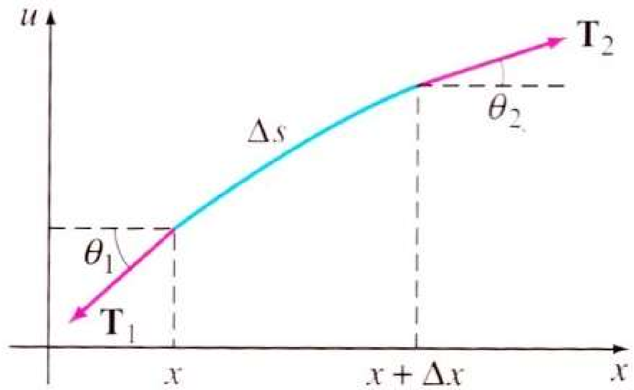
$$u_x(0, t) = u_x(L, t) = 0, \text{ for all } t.$$

- Physical intuition tells us that if the initial condition $f(x)$ is a reasonable function, there exists a unique solution $u(x, t)$ for the boundary value problem.



Derivation of the Wave Equation (1/2)

- A PDE that models the vibrations of a string can be derived with the following assumptions:
  - A perfectly flexible uniform string with density ρ is stretched under a uniform tension force of T between $x = 0$ and $x = L$.
  - Each point on the string moves only in u direction $\rightarrow u(x, t)$ is the shape of the string at time t.
  - The slope of the curve is small for all $x \rightarrow \sin\theta \approx \tan\theta = u_x(x, t)$.



Derivation of the Wave Equation (2/2)

- Apply Newton's law to the segment $[x, x + \Delta x]$,

$$
\begin{aligned}
T\sin\theta_2 - T\sin\theta_1 \approx T\tan\theta_2 - T\tan\theta_1 \\
= T[u_x(x + \Delta x, t) - u_x(x, t)] \\
= (\rho\Delta x)u_{tt}.
\end{aligned}
$$

- So, division by $\Delta x T$ on both side yields

$$\frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x} = \frac{\rho}{T} u_{tt}.$$

As $\Delta x \to 0$, we have $u_{xx} = (\rho/T)u_{tt}$.

BVP of the Wave Equation

- If we set

$$a^2 = \frac{T}{\rho},$$

we have the one-dimensional wave equation that models the free vibrations of a uniform flexible string:

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad (0 < x < L, t > 0);$$

$$u(0,t) = u(L,t) = 0, (t > 0),$$

$$u(x,0) = f(x) \qquad (0 < x < L),$$

$$u_t(x,0) = g(x) \qquad (0 < x < L).$$

Laplacian of a 2-D Function $u(x, y)$

- The Laplacian of the function $u(x, y)$ is defined as

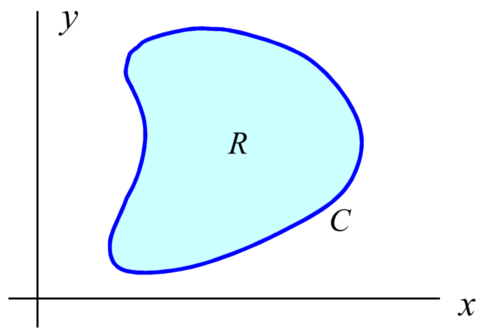
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

- The Laplace's equation $\nabla^2 u = 0$ is often used to model the steady-state behavior of a 2-D (or higher dimensional) phenomenon (e.g., temperature of an object).

Modeling of 2-D Heat/Wave Equations

- Given a 2-D thin plate with thermal diffusivity k, its temperature $u(x, y, t)$ at the point (x, y) at time t satisfies the 2-D heat equation:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = k\nabla^2 u, \quad k = \frac{K}{c\delta}$$



- Note that $u_t = k\nabla^2 u$ is the 2-D extension of the 1-D heat equation $u_t = ku_{xx}$. Similarly, $u_{tt} = a^2\nabla^2 u$ is the 2-D extension of the 1-D wave equation $u_{tt} = a^2 u_{xx}$.

Heat/Wave Eqs with Influences

- The 1-D heat/wave equation can be modified to take into account external and internal influences:

$$k\frac{\partial^2 u}{\partial x^2} + G(x,t,u,u_x) = \frac{\partial u}{\partial t},$$

and

$$a^2\frac{\partial^2 u}{\partial x^2} + F(x,t,u,u_t) = \frac{\partial^2 u}{\partial t^2},$$

where $G()$ may be the ambient temperature influences to the heated rod; and $F()$ may represent the external, damping, and restoring forces of the string vibration

Solution to the BVP of Heat Equation

- Note that the BVP of a heated rod is modelled as:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L,\ t > 0);$$

$$u(0,t) = u(L,t) = 0 \qquad (t > 0),$$

$$u(x,0) = f(x) \qquad (0 < x < L).$$

- Note that the heat equation is linear. That is, if u_1 and u_2 satisfy the PDE, $w = c_1u_1 + c_2u_2$ also satisfies the PDE.

  However, a solution of the PDE must also satisfy the boundary conditions.

Meeting Boundary Conditions (1/2)

- If u_1 and u_2 satisfies the (homogeneous) conditions

$$u(0, t) = u(L, t) = 0, \text{ for all } t > 0,$$

$w = c_1u_1 + c_2u_2$ will also satisfy the condition. However, the general form of w may not satisfy the boundary condition → only a particular choice of c_1 and c_2 satisfy the non-homogeneous boundary condition:

$$u(x, 0) = f(x), \quad 0 < x < L.$$

Meeting Boundary Conditions (2/2)

- In general, we must find an infinite sequence u_1, u_2, u_3, …, of solutions that satisfies both the PDE and the homogeneous boundary conditions, and assume the general solution form as follows:

$$u(x,t)=\sum_{n=1}^{\infty}c_n u_n(x,t).$$

Then, determine the coefficients c_1, c_2, c_3, … that satisfy the non-homogeneous boundary condition.

General Solutions of a Linear BVP

- Suppose that each of the functions u_1, u_2, u_3, …, satisfies both the PDE for $0 < x < L$ and $t > 0$ and the homogeneous conditions, and c_1, c_2, c_3, … are chosen to meet the following three criteria:
  1. For $0 < x < L$ and $t > 0$, the function $u(x, t) = \sum c_n u_n(x, t)$ is continuous and term-wise differentiable (for $\partial/\partial t$ and $\partial^2/\partial x^2$).
  2. $\sum_{n=1}^{\infty} c_n u_n(x,0) = f(x)$ for $0 < x < L$.
  3. The function $u(x, t) = \sum c_n u_n(x, t)$ is continuous within, and at the boundary of the region $0 \le x \le L$ and $t \ge 0$.

  Then $u(x, t)$ is a solution of the BVP.

Separation of Variables (1/4)

- In solving the heated rod problem, Fourier sought for a sequence of solutions u_1, u_2, u_3, …, which are "separable." That is for each of u_i, we have

$$u(x, t) = X(x)T(t),$$

where $X(x)$ and $T(t)$ are functions of x and t, respectively. Substitution of such $u(x, t)$ into the heat equation $u_t = ku_{xx}$ yields $XT' = kX''T$, or

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda,$$

where λ is a constant because changing x (or t) does not change T'/kT (or X''/X).

Separation of Variables (2/4)

- Thus, the solution can be obtained by solving two ODEs for some common value of λ:

$$X''(x) + \lambda X(x) = 0,$$
$$T'(t) + \lambda k T(t) = 0.$$

For $X(x)$, we have $u(0, t) = X(0)T(t) = 0$, $u(L, t) = X(L)T(t) = 0$.
Thus $X(0) = X(L) = 0$ if $T(t)$ is nontrivial.
$X(x)$ has a nontrivial solution if and only if

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1,2,3,...$$

and then

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1,2,3,...$$

Separation of Variables (3/4)

- To solve for $T(t)$, substituting the value λ into the ODE for $T(t)$ as

$$T_n' + \frac{n^2\pi^2 k}{L^2} T_n = 0.$$

A nontrivial solution of $T_n(t)$ is

$$T_n(t) = \exp\left(-n^2\pi^2 kt / L^2\right), \quad n = 1,2,3,\ldots$$

Separation of Variables (4/4)

- Now, we have sequences of solutions to the PDE

$$u_n(x, t) = X(x)T(t) = \exp(-n^2\pi^2 kt/L^2)\sin(n\pi x/L),$$

$n = 1, 2, 3, \ldots$. Each of these functions satisfies the heat equation and the homogeneous conditions.
We want to find $c_1, c_2, c_3, \ldots$ such that $\sum c_n u_n(x, t)$ satisfies

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = f(x), \quad 0 < x < L.$$

But this is the Fourier series of $f(x)$ on $[0, L]$. Thus,

$$c_n = b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1,2,3,\ldots.$$

Insulated Endpoint Conditions

- When the heated rod is insulated at both ends, the homogeneous boundary condition becomes $u_x(0, t) = u_x(L, t) = 0$. We can use the separation of variables approach again to solve this problem.

  Solving the ODE of $X(x)$ gives us:

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \cos\frac{n\pi x}{L}.$$

  Similarly, solving the ODE of $T(t)$ gives us:

$$T_n(t) = \exp\left(\frac{-n^2\pi^2 kt}{L^2}\right).$$

Heated Rod with Insulated Ends

- For a heated rod with zero endpoint temperatures, the general solution is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-n^2\pi^2 kt / L^2\right)\cos\frac{n\pi x}{L},$$

where $\{a_n\}$ are the Fourier cosine coefficients of $u(x, 0)$.

Solution to the BVP of Wave Equation

- The BVP of a vibrating string is modelled as:

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad (0 < x < L,\, t > 0);$$

$$u(0,t) = u(L,t) = 0,\ (t > 0),$$

$$u(x,0) = f(x) \qquad (0 < x < L),$$

$$u_t(x,0) = g(x) \qquad (0 < x < L).$$

Here, we have two non-homogeneous boundary conditions.

Problems with Two Nonzero BCs

- To solve the wave equation, we divide the system into two sub-problems:
  - Problem A:
    - $u_{tt} = a^2 u_{xx}$; $u(0, t) = u(L, t) = 0$, $u(x, 0) = f(x)$, $u_t(x, 0) = 0$.

      nonzero initial offset
  - Problem B:
    - $u_{tt} = a^2 y_{xx}$; $u(0, t) = u(L, t) = 0$, $u(x, 0) = 0$, $u_t(x, 0) = g(x)$.

      nonzero initial velocity

The overall solution is the sum of the two sub-problems since

$$u(x, 0) = u_A(x, 0) + u_B(x, 0) = f(x) + 0 = f(x),$$
$$u_t(x, 0) = \{u_A\}_t(x, 0) + \{u_B\}_t(x, 0) = 0 + g(x) = g(x).$$

Problem A Solution (1/3)

- By separation of variables, substitution of $u(x, t) = X(x)T(t)$ in $u_{tt} = a^2 u_{xx}$ yields $XT'' = a^2 X''T$ for all x and t. Therefore, assume that

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda, \text{ for some } \lambda.$$

$\rightarrow$ we have a system of ODE:

$$\begin{cases} X'' + \lambda X = 0, & X(0) = X(L) = 0 \\ T'' + \lambda a^2 T = 0, & T'(0) = 0 \end{cases}.$$

The first equation is an eigenvalue problem:

$$\lambda_n = \frac{n^2 \pi^2}{L^2}, n = 1,2,3,\ldots \quad \text{and} \quad X_n(x) = \sin\frac{n\pi x}{L}, n = 1,2,3,\ldots$$

Problem A Solution (2/3)

- Substitute λ_n into the second equation:

$$T_n'' + \frac{n^2\pi^2 a^2}{L^2} T_n = 0,\ \ T_n'(0) = 0.$$

  The solution to the IVP is $T_n(t) = A_n \cos\frac{n\pi at}{L},\ \ n = 1,2,3,\ldots$

- Hence,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}$$

  satisfies all the homogeneous boundary conditions.

  $\rightarrow$ Choose $\{A_n\}$ to satisfy the non-homogeneous boundary condition

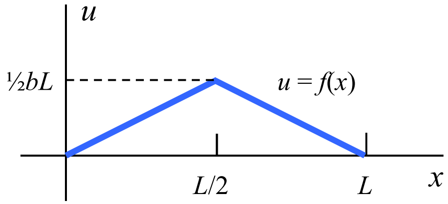
$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x),\ \ 0 < x < L.$$

Problem A Solution (3/3)

- If we choose $A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$

  the condition is simply the Fourier sine series expansion of $f(x)$ on $[0, L]$.

- Example: if $f(x) = \begin{cases} bx, & 0 \le x \le L/2 \\ b(L-x), & L/2 \le x \le L \end{cases},$



  and $g(x) = 0$, the solution $u(x, t)$ is

$$u(x,t) = \frac{4bL}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sin\frac{n\pi}{2}\right)\cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}.$$

d'Alembert form of Solution (1/2)

- An alternative form of solution of problem A can be obtained by applying trigonometric identity:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x+at) + \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x-at).$$

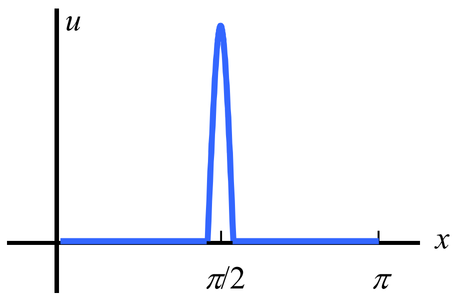
If we define $F(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L},$

we have

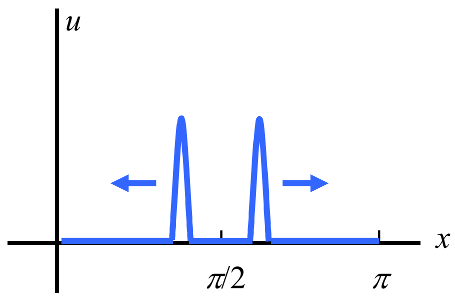
$$u(x, t) = [F(x + at) + F(x - at)]/2.$$

d'Alembert form of Solution (2/2)

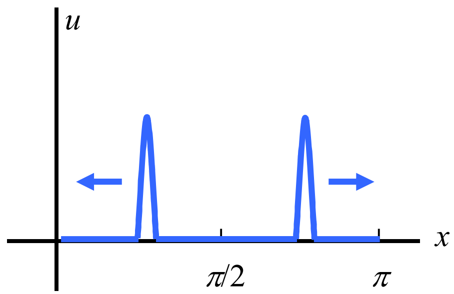
- The functions $F(x + at)$ and $F(x - at)$ in d'Alembert form of Solution represents waves moving to the left and right, respectively, along the string with speed a.



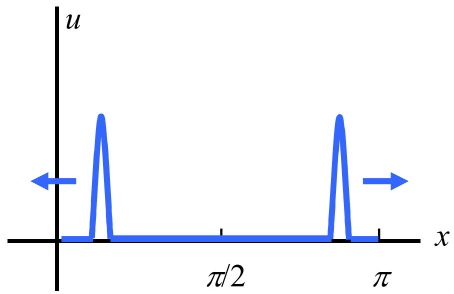
(a) At time $t = 0$.



(a) At time $t = \pi/8$.



(a) At time $t = \pi/4$.



(a) At time $t = 3\pi/8$.

Problem B Solution (1/2)

- Solution for Problem B is similar to that for A, except that

$$\frac{d^2T_n}{dt^2}+\frac{n^2\pi^2a^2}{L^2}T_n=0,\ T_n(0)=0.$$

A non-trivial solution is

$$T_n(t)=B_n\sin\frac{n\pi at}{L},\ n=1,2,3,\ldots$$

Hence,

$$u(x,t)=\sum_{n=1}^{\infty}X_n(x)T_n(t)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi at}{L}\sin\frac{n\pi x}{L}.$$

Problem B Solution (2/2)

- Again, the coefficients $\{B_n\}$ that satisfies the non-homogeneous boundary condition

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{L} \sin\frac{n\pi x}{L} = g(x),\ \ 0 < x < L.$$

would be the Fourier sine coefficient b_n of $g(x)$ on $[0, L]$ divided by $n\pi a/L$:

$$B_n \frac{n\pi a}{L} = b_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx.$$

Hence, we choose

$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}dx.$$

Total Solution to the Wave Equation

- The complete solution is the summation of Problem A and Problem B:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi at}{L} + B_n \sin \frac{n\pi at}{L} \right) \sin \frac{n\pi x}{L},$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx,$$

$$B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi x}{L} dx.$$

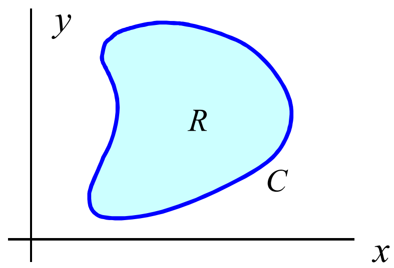
Steady-State Temperature

- The steady-state temperature of a plate can be described by a function $u(x, y)$, i.e., $u_t = 0$. Thus, we have the 2-D Laplace equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

- A boundary value problem of the Laplace equation can be formulated as follows (i.e. the Dirichlet problem):

$$\begin{cases} \nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & \text{(within R)} \\ u(x, y) = f(x, y) & \text{(if } (x, y) \text{ is on } C\text{)} \end{cases}.$$



Solutions to the Laplace's Equation

- Suppose we want to find the steady-state temperature $u(x, y)$ in a thin rectangular plate with width a and height b. The problem can be formulated as a BVP problem as follows:

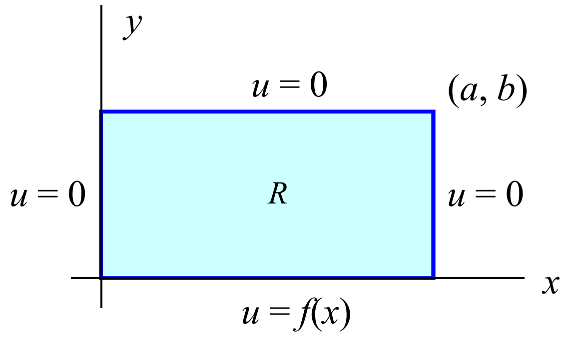
$$u_{xx} + u_{yy} = 0;$$
$$u(0, y) = f_1(x),\ u(a, y) = f_2(x),\ u(x, b) = f_3(x),\ u(x, 0) = f_4(x).$$

This is called the Dirichlet problem.

Example: The Dirichlet Problem (1/4)

- Solve the boundary value problem for the rectangle R.

$$u_{xx} + u_{yy} = 0;$$
$$u(0, y) = u(a, y) = u(x, b) = 0,$$
$$u(x, 0) = f(x).$$



Assume that $u(x, y) = X(x)Y(y)$, we have $X''Y + XY'' = 0$. Thus,

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \quad \rightarrow \quad \begin{cases} X'' + \lambda X = 0 \\ X(0) = X(a) = 0 \end{cases}.$$

Example: The Dirichlet Problem (2/4)

- The eigenvalues and eigenfunctions of X are

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \quad X_n(x) = \sin\frac{n\pi x}{a}, \quad n = 1,2,3,\ldots$$

As a result,

$$Y_n'' - \frac{n^2\pi^2}{a^2} Y_n = 0, \quad Y_n(b) = 0.$$

The general solution of Y_n is

$$Y_n(y) = A_n \cosh\frac{n\pi y}{a} + B_n \sinh\frac{n\pi y}{a}.$$

Example: The Dirichlet Problem (3/4)

- To compute the particular solution, we must solve A_n and B_n using $Y_n(b) = 0$:

$$Y_n(b) = A_n \cosh\frac{n\pi b}{a} + B_n \sinh\frac{n\pi b}{a} = 0.$$

$$\rightarrow \quad B_n = -A_n \cosh\frac{n\pi b}{a} \Big/ \sinh\frac{n\pi b}{a}.$$

Therefore,

$$Y_n(y) = A_n \cosh\frac{n\pi y}{a} - \left(A_n \cosh\frac{n\pi b}{a} \Big/ \sinh\frac{n\pi b}{a} \right) \sinh\frac{n\pi y}{a}$$

$$= c_n \sinh\frac{n\pi(b-y)}{a}, \;\; c_n = A_n / \sinh(n\pi b / a).$$

Example: The Dirichlet Problem (4/4)

- The formal series solution is then

$$u(x,y)=\sum_{n=1}^{\infty}X_n(x)Y_n(y)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi(b-y)}{a}.$$

c_n must satisfy the nonhomogeneous condition

$$u(x,0)=\sum_{n=1}^{\infty}\left(c_n\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a}=f(x).$$

Therefore,

$$c_n=\frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx.$$

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