The Laplace Transform†

National Chiao Tung University
Chun-Jen Tsai
10/23/2019

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† Chapter 7 in the textbook.

Transform of a Function

- ❑ Some operators transform a function into another function:

Differentiation: $\frac{d}{dx}x^2 = 2x$, or $Dx^2 = 2x$

Indefinite Integration: $\int x^2 dx = \frac{x^3}{3} + c$

Definite Integration: $\int_0^3 x^2 dx = 9$

$\rightarrow$ A function may have nicer property in the transformed domain!

Integral Transform

- If $f(x, y)$ is a function of two variables, then a definite integral of f w.r.t. one of the variable leads to a function of the other variable.

  Example: $\int_1^2 2xy^2 dx = 3y^2$

- Improper integral of a function defines how integration can be calculated over an infinite interval:

$$\int_0^\infty K(s,t) f(t) dt \equiv \lim_{b\to\infty} \int_0^b K(s,t) f(t) dt$$

Laplace Transform

- ☐ **Definition:** Let f be a function defined for $t \geq 0$, then the integral

$$\mathscr{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$

is said to be the **Laplace transform** of f, provided the integral converges.

The result of the Laplace transform is a function of s, usually referred to as $F(s)$.

Example: $\mathscr{L}\{1\}$

- By definition:

$$\mathscr{L}(1) = \int_0^\infty e^{-st}(1)dt = \lim_{b\to\infty}\int_0^b e^{-st}dt$$

$$= \lim_{b\to\infty} \left.\frac{-e^{-st}}{s}\right|_0^b = \lim_{b\to\infty}\frac{-e^{-sb}+1}{s} = \frac{1}{s},$$

provided $s > 0$. The integral diverges for $s < 0$.

Example: $\mathscr{L}\{t\}$

- ☐ By definition: $\mathscr{L}(t) = \int_0^\infty e^{-st} t dt$

  Using integration by parts and apply l'Hospital's rule to get $\lim_{t\to\infty} te^{-st} = 0$, $s > 0$, we have:

$$\mathscr{L}\{t\} = \left.\frac{-te^{-st}}{s}\right|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt$$

$$= \frac{1}{s}\mathscr{L}\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

Example: $\mathscr{L}\{e^{at}\}$

- ☐ By definition:

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt$$

$$= \left. \frac{-e^{-(s-a)t}}{s-a} \right|_0^\infty$$

$$= \frac{1}{s-a}, s > a$$

Example: $\mathscr{L}\{t^n\}, n \in N$

- Similarly, let $u = t^n$, $dv = e^{-st}dt$,

$$\mathscr{L}\{t^n\} = \int_0^\infty e^{-st} t^n dt = -\frac{t^n}{s} e^{-st}\Bigg|_0^\infty + \int_0^\infty \frac{n}{s} e^{-st} t^{n-1} dt$$

$$= \frac{n}{s}\mathscr{L}\left\{t^{n-1}\right\} = \frac{n(n-1)}{s^2}\mathscr{L}\left\{t^{n-2}\right\} = \ldots = \frac{n(n-1)\cdots 2}{s^{n-1}}\mathscr{L}\{t\}$$

$$= \frac{n!}{s^{n+1}}, s > 0.$$

Example: $\mathscr{L}\{\sin 2t\}$

- By definition:

$$\mathscr{L}\{\sin 2t\} = \int_0^\infty e^{-st}\sin 2t\,dt = \left.\frac{-e^{-st}\sin 2t}{s}\right|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt$$

$$= \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt\ ,\ s>0$$

$$= \frac{2}{s}\left[\left.\frac{-e^{-st}\cos 2t}{s}\right|_0^\infty - \frac{2}{s}\int_0^\infty e^{-st}\sin 2t\,dt\right]$$

$$= \frac{2}{s^2} - \frac{4}{s^2}\mathscr{L}\{\sin 2t\} \to \mathscr{L}\{\sin 2t\} = \frac{2}{s^2+4}, s>0$$

Linearity of $\mathscr{L}\{\}$

- For a sum of functions, we can write

$$\int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]dt = \alpha \int_0^\infty e^{-st} f(t)dt + \beta \int_0^\infty e^{-st} g(t)dt$$

whenever both integrals converge for $s > c$, where c is some constant.

Hence,

$$\begin{aligned} &\mathscr{L}\{\alpha f(t) + \beta g(t)\} \\ &= \alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\} \\ &= \alpha F(s) + \beta G(s) \end{aligned}$$

Example: $\mathscr{L}\{3e^{2t} + 2\sin^2 3t\}$

- ☐ $\mathscr{L}\{3e^{2t} + 2\sin^2 3t\} = 3\mathscr{L}\{e^{2t}\} + \mathscr{L}\{2\sin^2 3t\}$

$$= 3/(s - 2) + \mathscr{L}\{1 - \cos 6t\}$$

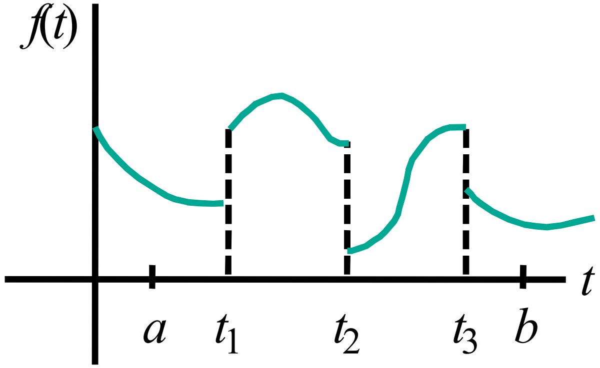
$$= 3/(s - 2) + [1/s - s/(s^2 + 36)],\ s > 2.$$

Transform of Basic Functions

- ☐ $\mathscr{L}\{1\} = \dfrac{1}{s}$
- ☐ $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a},\ s > a$
- ☐ $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}},\ \ n = 1, 2, 3, \cdots$
- ☐ $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2 + k^2}$
- ☐ $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2 + k^2}$
- ☐ $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2 - k^2}$
- ☐ $\mathscr{L}\{\sinh kt\} = \dfrac{k}{s^2 - k^2}$

Existence of $\mathscr{L}\{f(t)\}$

- **Theorem:** If f is piecewise continuous on $[0, \infty)$, and that f is of exponential order for $t > T$, where T is a constant, then $\mathscr{L}\{f(t)\}$ converges.

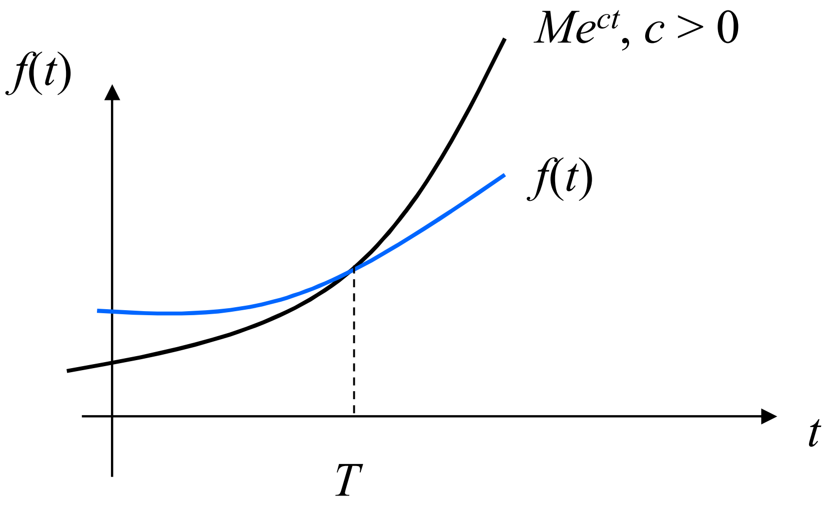


- **Definition:** A function f is said to be of exponential order c if there exists constants c, $M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$.

Examples: Exponential Order

- ☐ The functions $f(t) = t$, $f(t) = e^{-t}$, and $f(t) = 2\cos t$ are all of exponential order $c = 1$ for $t > 0$, since we have

  $| t | \leq e^t$, $| e^{-t} | \leq e^t$, $| 2\cos t | \leq 2e^t$.



Proof of Existence of $\mathscr{L}\{f(t)\}$

- By the additive interval property of definite integrals,

$$\mathscr{L}\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt = I_1 + I_2$$

The integral I_1 exists (finite interval, f piecewise continuous). Now,

$$|I_2| \le \int_T^\infty |e^{-st} f(t)|\,dt \le M \int_T^\infty e^{-st} e^{ct} dt$$

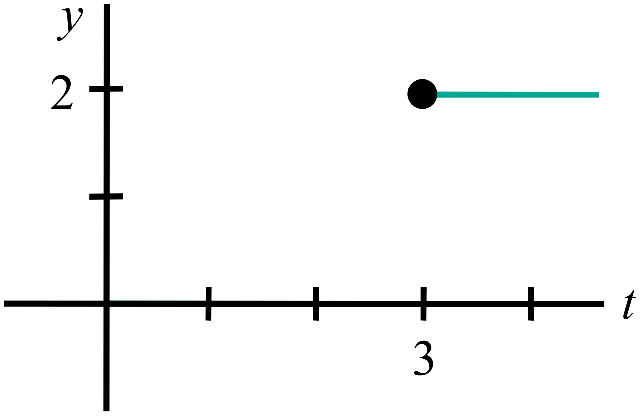
$$= M \int_T^\infty e^{-(s-c)t} dt = -M \left. \frac{e^{-(s-c)t}}{s-c} \right|_T^\infty = M \frac{e^{-(s-c)T}}{s-c}, \quad s > c.$$

→ I_2 exists as well → $\mathscr{L}\{f(t)\}$ converges.

Example: Transform of Piecewise $f(t)$

- Evaluate $\mathscr{L}\{f(t)\}$ for

$$f(t)=\begin{cases}0, & 0\le t<3\\ 2, & t\ge 3.\end{cases}$$



Solution:

$$\mathscr{L}\{f(t)\}=\int_0^{\infty}e^{-st}f(t)dt$$

$$=\int_0^{\infty}e^{-st}f(t)dt=\int_0^{3}(e^{-st}\cdot 0)dt+\int_3^{\infty}2e^{-st}dt$$

$$=-\frac{2e^{-st}}{s}\bigg|_3^{\infty}=\frac{2e^{-3s}}{s},\quad s>0.$$

Behavior of $F(s)$ as $s \to \infty$

- If f is piecewise continuous on $[0, \infty)$ and of exponential order for $t > T$, then $\lim_{s\to\infty}\mathscr{L}\{f(t)\} = 0$.

  Proof:

  Since $f(t)$ is piecewise continuous on $0 \le t \le T$, it is necessarily bounded on the interval. That is $|f(t)| \le M_1 e^{0t}$. Also, $|f(t)| \le M_2 e^{\gamma t}$ for $t > T$. If M denotes the maximum of $\{M_1, M_2\}$ and c denotes the maximum of $\{0, \gamma\}$, then for $s > c$:

$$\mathscr{L}\{f(t)\} \le \int_0^\infty e^{-st}\,|f(t)|\,dt \le M\int_0^\infty e^{-st}\cdot e^{ct}dt$$

$$= -M\left.\frac{e^{-(s-c)t}}{s-c}\right|_0^\infty = \frac{M}{s-c} \to 0 \text{ as } s \to \infty.$$

Inverse Laplace Transform

- ☐ If $F(s)$ is the Laplace transform of a function $f(t)$, namely, $\mathscr{L}\{f(t)\} = F(s)$, then we say that $f(t)$ is the inverse Laplace transform of $F(s)$, that is,

$$f(t) = \mathscr{L}^{-1}\{F(s)\}.$$

- ☐ Example:

$1 = \mathscr{L}^{-1}\{1/s\}$, $t = \mathscr{L}^{-1}\{1/s^2\}$, and $e^{-3t} = \mathscr{L}^{-1}\{1/(s+3)\}$.

Examples: Inverse Transforms

- ❑ Evaluate $\mathscr{L}^{-1}\{1/s^5\}$

  Solution:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}t^4$$

- ❑ Evaluate $\mathscr{L}^{-1}\{1/(s^2 + 7)\}$

  Solution:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\mathscr{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin\sqrt{7}t$$

Linearity of $\mathscr{L}^{-1}\{\}$

- The inverse Laplace transform is also a linear transform; that is, for constant α and β,

$$\mathscr{L}^{-1}\{\alpha F(s)+\beta G(s)\}=\alpha\mathscr{L}^{-1}\{F(s)\}+\beta\mathscr{L}^{-1}\{G(s)\}$$

- Example: Evaluate $\mathscr{L}^{-1}\{(-2s+6)/(s^2+4)\}$

$$\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}=\mathscr{L}^{-1}\left\{\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right\}$$

$$=-2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{6}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$=-2\cos 2t+3\sin 2t$$

Example: Partial Fractions (1/2)

- Evaluate

$$\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

Solution:

There exists unique constants A, B, C such that:

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}=\frac{A}{(s-1)}+\frac{B}{(s-2)}+\frac{C}{(s+4)}$$

$$=\frac{A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)}{(s-1)(s-2)(s+4)}$$

By comparing terms, we have

Example: Partial Fractions (2/2)

Partial fractions:

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{\frac{16}{5}}{(s-1)} + \frac{\frac{25}{6}}{(s-2)} + \frac{\frac{1}{30}}{(s+4)}$$

Therefore

$$\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

$$= -\frac{16}{5}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

Partial Fraction Decompositions

- Inverse Laplace transform usually involves partial fractions decomposition, let $P(s)$ be a polynomial function with degree less than n:
  - Linear factor decomposition:

$$\frac{P(s)}{(s-a)^n}=\frac{A_1}{s-a}+\frac{A_2}{(s-a)^2}+...+\frac{A_n}{(s-a)^n},$$

  where A_1, A_2, …, A_n are constants.

  - Quadratic factor decomposition

$$\frac{P(s)}{[(s-a)^2+b^2]^n}=\frac{A_1s+B_1}{(s-a)^2+b^2}+\frac{A_2s+B_2}{[(s-a)^2+b^2]^2}+...+\frac{A_ns+B_n}{[(s-a)^2+b^2]^n},$$

  where A_1, …, A_n and B_1, …, B_n are constants.

Transforming a Derivative

- ☐ What is the Laplace transform of $f'(t)$?

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$= -f(0) + s\mathscr{L}\{f(t)\}$$

Therefore

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

→ Note that this derivation only works if $f'(t)$ is a continuous function

Derivative Transform Theorem

- **Theorem:** If the function $f(t)$ is continuous and piecewise smooth for $t \geq 0$ and is of exponential order as $t \to +\infty$, so that there exist nonnegative constants M, c, and T such that

$$|f(t)| \leq Me^{ct} \quad \text{for } t \geq T.$$

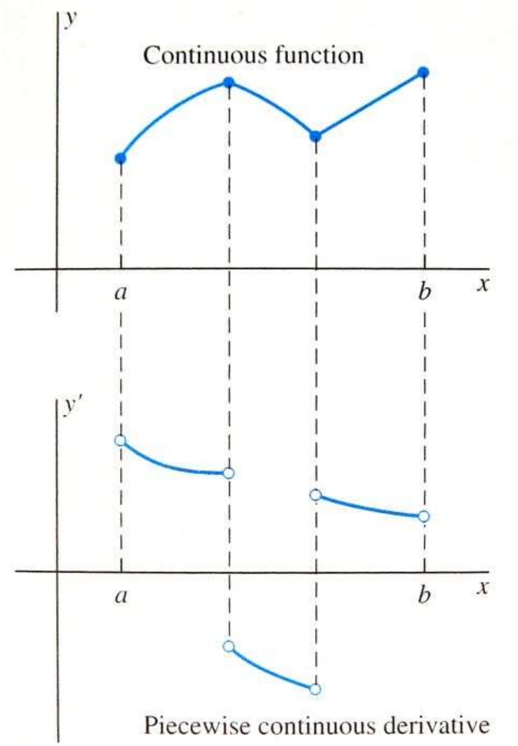
Then $\mathscr{L}\{f'(t)\}$ exists for $s > c$, and

$$\mathscr{L}\left\{f'(t)\right\} = sF(s) - f(0).$$

Proof:

Perform (finite) piece-by-piece integration of

$$\int_0^\infty e^{-st} f'(t)dt$$



General Derivative Transform

- **Theorem:** If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

Solving Linear IVPs (1/2)

- ❑ The Laplace transform of a linear DE with constant coefficients becomes an algebraic equation in $X(s)$. That is,

$$\mathscr{L}\left\{a_n \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_0 x\right\} = \mathscr{L}\{f(t)\}$$

becomes

$$a_n\mathscr{L}\left\{\frac{d^n x}{dt^n}\right\} + a_{n-1}\mathscr{L}\left\{\frac{d^{n-1}x}{dt^{n-1}}\right\} + \cdots + a_0\mathscr{L}\{x\} = \mathscr{L}\{f(t)\},$$

or

$$a_n[s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \ldots - x^{(n-1)}(0)]$$
$$+ a_{n-1}[s^{n-1}X(s) - s^{n-2}x(0) - \ldots - x^{(n-2)}(0)] + \ldots + a_0 X(s) = F(s)$$

Solving Linear IVPs (2/2)

- ☐ Given initial conditions $x(0) = x_0, x'(0) = x_1, \ldots, x^{(n-1)}(0) = x_{n-1}$, we have $Z(s)X(s) = I(s) + F(s)$, or

$$X(s) = \underbrace{\frac{I(s)}{Z(s)}}_{\text{transient behavior}} + \underbrace{\frac{F(s)}{Z(s)}}_{\text{steady state behavior}}$$

where $Z(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0$ and

$$\begin{aligned} I(s) = {} & (a_n s^{n-1} + a_{n-1}s^{n-2} + \ldots + a_1)x(0) \\ & +(a_n s^{n-2} + a_{n-1}s^{n-3} + \ldots + a_2)x'(0) \\ & + \ldots + a_n x^{(n-1)}(0). \end{aligned}$$

Example: $\dfrac{dy}{dt}+3y=13\sin 2t$, $y(0)=6$ (1/2)

- Since

$$\mathscr{L}\left\{\frac{dy}{dt}\right\}+3\mathscr{L}\{y\}=13\mathscr{L}\{\sin 2t\},$$

$\mathscr{L}\{dy/dt\} = sY(s) - y(0) = sY(s) - 6$, and $\mathscr{L}\{\sin 2t\} = 2/(s^2 + 4)$, we have

$$sY(s)-6+3Y(s)=\frac{26}{s^2+4},$$

or

$$(s+3)Y(s)=6+\frac{26}{s^2+4},$$

$$\rightarrow\ Y(s)=\frac{6}{(s+3)}+\frac{26}{(s+3)(s^2+4)}=\frac{6s^2+50}{(s+3)(s^2+4)}.$$

Example: $\dfrac{dy}{dt}+3y=13\sin 2t$, $y(0)=6$ (2/2)

Assume that

$$\frac{6s^2+50}{(s+3)(s^2+4)}=\frac{A}{s+3}+\frac{Bs+C}{s^2+4},$$

we have $A = 8$, $B = –2$, $C = 6$. Therefore

$$y(t)=8\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\}-2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+3\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$\rightarrow y(t)=8e^{-3t}-2\cos 2t+3\sin 2t$$

Example: $y'' - 3y' + 2y = e^{-4t}$, $y(0) = 1$, $y'(0) = 5$

- Solution:

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathscr{L}\left\{\frac{dy}{dt}\right\} + 2\mathscr{L}\{y\} = \mathscr{L}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$Y(s) = \frac{s+2}{s^2 - 3s + 2} + \frac{1}{(s^2 - 3s + 2)(s+4)}$$

$$= \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)}$$

$$\rightarrow y(t) = \mathscr{L}^{-1}\{Y(s)\} = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

s-axis Translation Theorem

- **Theorem:** If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathscr{L}\{e^{at}f(t)\} = F(s - a).$$

Proof:

$$\mathscr{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t) dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t) dt, \ \ s > a$$

$$= F(s-a), \ \ s > a.$$

Example: $\mathscr{L}\{e^{5t}t^3\}$ and $\mathscr{L}\{e^{-2t}\cos 4t\}$

- ❑ Solution:

$$\mathscr{L}\{e^{5t}t^3\} = \mathscr{L}\{t^3\}_{s\to s-5} = \left.\frac{3!}{s^4}\right|_{s\to s-5} = \frac{6}{(s-5)^4}$$

$$\mathscr{L}\{e^{-2t}\cos 4t\} = \mathscr{L}\{\cos 4t\}_{s\to s-(-2)}$$

$$= \left.\frac{s}{s^2+16}\right|_{s\to s+2} = \frac{s+2}{(s+2)^2+16}$$

Inverse of *s*-axis Translation

- ☐ The inverse Laplace transform of $F(s-a)$, can be computed multiplying $f(t) = \mathscr{L}^{-1}\{F(s)\}$ by e^{at}:

$$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\{F(s)\big|_{s\to s-a}\} = e^{at} f(t)$$

- ☐ Example: Compute $\mathscr{L}^{-1}\{(2s+5)/(s-3)^2\}$.

Since $$\frac{2s+5}{(s-3)^2} = \left.\frac{2s+11}{s^2}\right|_{s\to s-3},$$

$$\to \mathscr{L}^{-1}\left\{\left.\frac{2s+11}{s^2}\right|_{s\to s-3}\right\} = 2\mathscr{L}^{-1}\left\{\left.\frac{1}{s}\right|_{s\to s-3}\right\} + 11\mathscr{L}^{-1}\left\{\left.\frac{1}{s^2}\right|_{s\to s-3}\right\}$$

$$= 2e^{3t} + 11e^{3t}t.$$

Example: $y'' - 6y' + 9y = t^2 e^{3t}$

- Solve the DE with initial conditions $y(0) = 2, y'(0) = 17$.

$$s^2 Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 9Y(s) = \frac{2!}{(s-3)^3}$$

$$Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}$$

$$= \left.\frac{2s+11}{s^2}\right|_{s\to s-3} + \left.\frac{2}{s^5}\right|_{s\to s-3}$$

$$y(t) = 2\mathscr{L}^{-1}\left\{\left.\frac{1}{s}\right|_{s\to s-3}\right\} + 11\mathscr{L}^{-1}\left\{\left.\frac{1}{s^2}\right|_{s\to s-3}\right\} + \frac{2}{4!}\mathscr{L}^{-1}\left\{\left.\frac{4!}{s^5}\right|_{s\to s-3}\right\}$$

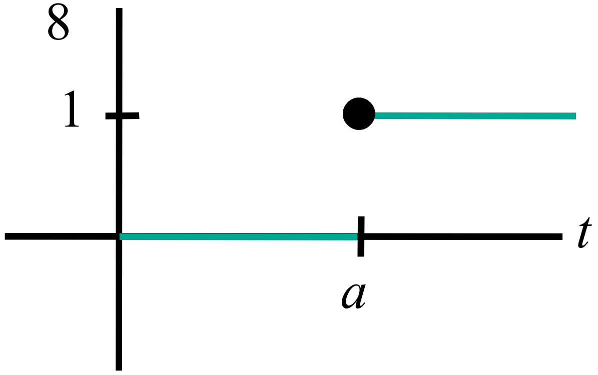
$$= 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4 e^{3t}.$$

Unit Step Function

- The unit step function $u(t-a)$ is defined to be

$$u(t-a)=\begin{cases}0\,, & 0\le t<a\\ 1\,, & t\ge a\end{cases}.$$

$u(t-a)$ is often denoted as $u_a(t)$. Note that $u_a(t)$ is only defined on the non-negative axis since the Laplace transform is only defined on this domain.



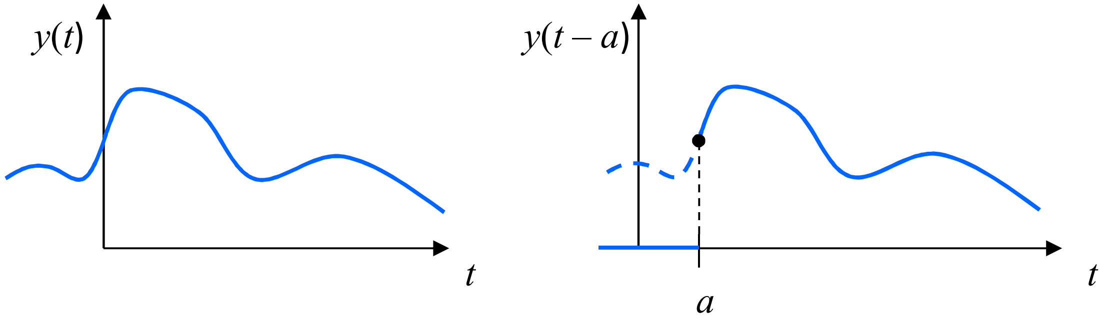
Rewrite of a Piecewise Function

- ☐ A piecewise defined function can be rewritten in a compact form using $u(t-a)$.

  For example,

$$f(t)=\begin{cases} g(t)\,, & 0\le t<a \\ h(t), & t\ge a \end{cases}$$

  is the same as $f(t)=g(t)-g(t)u(t-a)+h(t)u(t-a)$.



Laplace Transform of $u(t-a)$, $a > 0$

- ☐ By definition,

$$\mathscr{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt = \int_a^\infty e^{-st} dt$$

$$= \lim_{b\to\infty} \left[-\frac{e^{-st}}{s} \right]_{t=a}^{b}.$$

Therefore,

$$\mathscr{L}\{u(t-a)\} = \frac{e^{-as}}{s} \; (s > 0,\ a > 0).$$

t-axis Translation Theorem

- **Theorem:** If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then $\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$.

  Proof:

$$\int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t-a)u(t-a)\,dt + \int_a^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Let $v = t - a$, $dv = dt$,

$$\mathscr{L}\left\{f(t-a)u(t-a)\right\} = \int_0^\infty e^{-s(v+a)} f(v)\,dv = e^{-as}\mathscr{L}\left\{f(t)\right\}$$

Inverse of *t*-axis Translation

- If $f(t) = \mathscr{L}^{-1}\{F(s)\}$ and $a > 0$, the inverse form of the t-axis translation theorem is:

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a).$$

- Example:

$$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s^3}\right\} = u(t-a)\frac{1}{2}(t-a)^2 = \begin{cases} 0, & \text{if } t < a \\ \frac{1}{2}\left(t-a\right)^2, & \text{if } t \geq a \end{cases}.$$

Alternative Form of *t*-axis Translation

- For $g(t)u(t-a)$, we can derive an alternative form:

$$\mathscr{L}\{g(t)u(t-a)\} = \int_a^\infty e^{-st} g(t)dt = \int_0^\infty e^{-s(v+a)} g(v+a)dv$$

$$= e^{-sa}\int_0^\infty e^{-sv} g(v+a)dv = e^{-as}\mathscr{L}\{g(t+a)\}$$

$$\rightarrow \mathscr{L}\{g(t)u(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$$

- Example: Since $g(t+\pi) = \cos\,(t+\pi) = -\cos\,t$,

$$\mathscr{L}\{\cos t\; u(t-\pi)\} = e^{-\pi s}\mathscr{L}\{-\cos t\} = -\frac{s}{s^2+1}e^{-\pi s}.$$

Example: $y' + y = f(t),\ y(0) = 5,\ f(t) = \begin{cases} 0\,, & 0 \le t < \pi \\ 3\cos t\,, & t \ge \pi \end{cases}$

- Note that $f(t) = 3\cos t\, u(t - \pi)$, we have
  $\mathscr{L}\{y'\} + \mathscr{L}\{y\} = 3\ \mathscr{L}\{\cos t\, u(t - \pi)\}$,

$$Y(s) = \frac{5}{s+1} - \frac{3}{2}\left[-\frac{1}{s+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-\pi s} + \frac{s}{s^2+1}e^{-\pi s} \right]$$

$$y(t) = 5e^{-t} + \left[\frac{3}{2}e^{-(t-\pi)} - \frac{3}{2}\sin(t-\pi) - \frac{3}{2}\cos(t-\pi) \right] u(t-\pi)$$

$$= 5e^{-t} + \left[\frac{3}{2}e^{-(t-\pi)} + \frac{3}{2}\sin(t) + \frac{3}{2}\cos(t) \right] u(t-\pi).$$

Derivatives of Transforms

- ☐ **Theorem:** If $f(t)$ is piecewise continuous and $f(t)$ is of exponential order, then

$$\mathscr{L}\left\{-tf(t)\right\} = \frac{d}{ds}\mathscr{L}\left\{f(t)\right\} = F'(s).$$

Proof:

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\right]dt$$

$$= \int_0^\infty -e^{-st} tf(t)dt = \mathscr{L}\left\{-tf(t)\right\}$$

#

Note:

$$\mathscr{L}\left\{tf(t)\right\} = -\frac{d}{ds}\mathscr{L}\left\{f(t)\right\} \rightarrow f(t) = -\frac{1}{t}\mathscr{L}^{-1}\left\{F'(s)\right\}$$

nth-Order Derivatives of Transforms

- **Theorem:** If $F(s) = \mathscr{L}\{f(t)\}$ and $n = 1,2,3\ldots$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Proof:

The proof can be done by mathematical induction. Here, we only check the 2nd-order case.

$$\mathscr{L}\left\{t^2 f(t)\right\} = \mathscr{L}\left\{t \cdot tf(t)\right\} = -\frac{d}{ds}\mathscr{L}\left\{tf(t)\right\} = \frac{d^2}{ds^2}\mathscr{L}\left\{f(t)\right\}$$

- Example: Compute $\mathscr{L}\{t \sin kt\}$.

$$\mathscr{L}\{t \sin kt\} = -\frac{d}{ds}\mathscr{L}\{\sin kt\} = -\frac{d}{ds}\left(\frac{k}{s^2+k^2}\right) = \frac{2ks}{(s^2+k^2)^2}$$

Convolution of Two Functions

- If f and g are piecewise continuous on $[0, \infty)$, then a special product, denoted by $f * g$, is defined by the integral

$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau$$

  and is called the convolution of f and g. The convolution is a function of t. Note that $f * g = g * f$.

- Example:

$$e^t * \sin t = \int_0^t e^\tau \sin(t-\tau)d\tau = \frac{1}{2}(-\sin t - \cos t + e^t).$$

Convolution Theorem

- **Theorem:** If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s).$$

Proof:

$$F(s)G(s) = \left(\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right)\left(\int_0^\infty e^{-s\beta} g(\beta)\,d\beta\right)$$

$$= \int_0^\infty f(\tau)\left(\int_0^\infty e^{-s(\tau+\beta)} g(\beta)\,d\beta\right)d\tau.$$

Let $t = \tau + \beta$, $dt = d\beta$, so that

$$F(s)G(s) = \int_0^\infty e^{-st}\left(\int_0^\infty f(\tau)g(t-\tau)\,d\tau\right)dt = \mathscr{L}\{f * g\}.$$

Example: Compute $\mathscr{L}\left\{\int_0^t e^{\tau}\sin(t-\tau)d\tau\right\}$

- Solution:

$$\mathscr{L}\left\{\int_0^t e^{\tau}\sin(t-\tau)d\tau\right\}=\mathscr{L}\left\{e^{t}\right\}\cdot\mathscr{L}\left\{\sin t\right\}$$

$$=\frac{1}{s-1}\cdot\frac{1}{s^2+1}=\frac{1}{(s-1)(s^2+1)}$$

Inverse Form of Convolution

- ☐ **Theorem:**

$$\mathscr{L}^{-1}\left\{F(s)G(s)\right\} = f * g.$$

- ☐ Example: $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+k^2)^2}\right\}$

Let $F(s) = G(s) = 1/(s^2 + k^2)$,

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{k^2}\int_0^t \sin k\tau \sin k(t-\tau)\,d\tau,$$

$$= \frac{1}{2k^2}\int_0^t \left[\cos k(2\tau - t) - \cos kt\right]d\tau = \frac{\sin kt - kt\cos kt}{2k^3}.$$

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† sin A sin B = [cos(A – B) – cos(A + B)]/2.

Transforms of Integrals

- **Theorem:** The Laplace transform of the integral of a piecewise continuous function $f(t)$ of exponential order is

$$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}.$$

The inverse form is:

$$\int_0^t f(\tau)d\tau = \mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\}.$$

(Recall that: $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$).

Proof of Transform of Integrals

- Since $f(t)$ is piecewise continuous, by fundamental theorem of calculus, if $g(t) = \int_0^t f(\tau)d\tau$, $g(t)$ is continuous and $g'(t) = f(t)$ where $f(t)$ is continuous.
  Because $f(t)$ is of exponential order, there exists constants M and c such that
  $$\left|g(t)\right| \le \int_0^t \left|f(\tau)\right| d\tau \le M\int_0^t e^{c\tau}d\tau = \frac{M}{c}(e^{ct}-1) < \frac{M}{c}e^{ct}.$$
  $\to g(t)$ is of exponential order as $t \to +\infty$.
  Thus, $\mathscr{L}\{f(t)\} = \mathscr{L}\{g'(t)\} = s\mathscr{L}\{g(t)\} - g(0)$.
  But $g(0) = 0$, therefore,
  $$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \mathscr{L}\left\{g(t)\right\} = \frac{F(s)}{s}.$$

Example: Inverse by Integration

- Starting with $f(t) = \sin t$, $F(s) = 1/(s^2+1)$ we have:

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin\tau \, d\tau = 1 - \cos t,$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos\tau) \, d\tau = t - \sin t,$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (\tau - \sin\tau) \, d\tau = \tfrac{1}{2} t^2 - 1 + \cos t.$$

Integral Equations

- ☐ We can use convolution theorem to solve differential equations as well as “integral equations”.

  For example, the Volterra integral equation:

$$f(t) = g(t) + \int_0^t f(\tau)h(t-\tau)\, d\tau,$$

  where $g(t)$ and $h(t)$ are known.

Example: $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau) e^{t-\tau} d\tau.$

- Solution: notice that $h(t) = e^t$. Take the Laplace transform of each term:

$$F(s) = 3 \cdot \frac{2}{s^3} - \frac{1}{s+1} - F(s) \cdot \frac{1}{s-1}$$

$$= \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}.$$

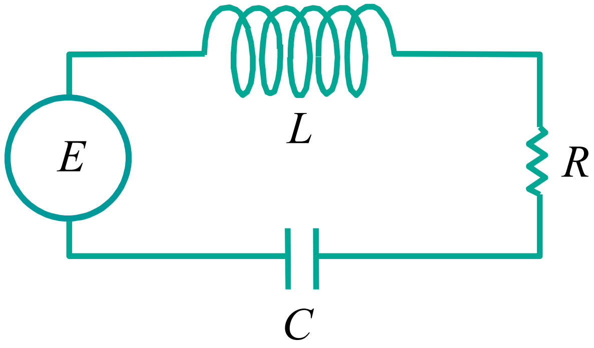
The inverse transform then gives:

$$f(t) = 3t^2 - t^3 + 1 - 2e^{-t}.$$

Series Circuits

- ☐ The current in a circuit is governed by the integrodifferential equation

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)\,d\tau=E(t).$$



Example: Single-loop LRC Circuit

- Given $L = 0.1\text{h}$, $R = 2\Omega$, $C = 0.1\text{f}$, $i(0) = 0$, and $E(t) = 120t - 120t\mathscr{U}(t - 1)$, find $i(t)$.

  Solution:

  Since $$0.1\frac{di}{dt} + 2i + 10\int_0^t i(\tau)\, d\tau = 120t - 120t\mathscr{U}(t-1),$$

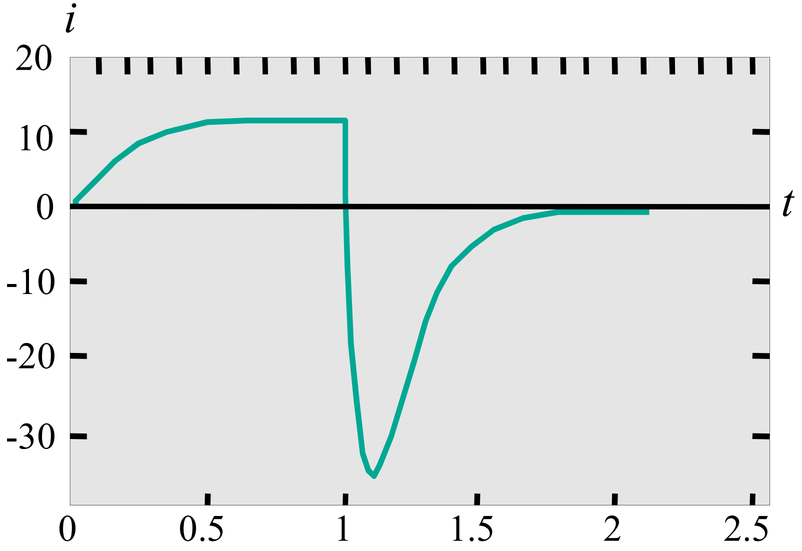
  and $\mathscr{L}\left\{\int_0^t i(\tau)\, d\tau\right\} = I(s)/s$, we have

  $$0.1sI(s) + 2I(s) + 10\frac{I(s)}{s} = 120\left[\frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right].$$

  $$\to I(s) = 1200\left[\frac{1}{s(s+10)^2} - \frac{1}{s(s+10)^2}e^{-s} - \frac{1}{(s+10)^2}e^{-s}\right].$$

Example: continued

$$\rightarrow \begin{cases} 12 - 12e^{-10t} - 120te^{-10t}, & 0 \le t < 1 \\ -12e^{-10t} + 12e^{-10(t-1)} - 120te^{-10t} - 1080(t-1)e^{-10(t-1)}, & t \ge 1 \end{cases}$$



Transform of a Periodic Function

- If a periodic function f has period T, $T > 0$, then $f(t + T) = f(t)$. The Laplace transform of a periodic function can be obtained by integration over one period.

- **Theorem:** If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$

Proof of Periodic Transform Theorem

- **Proof:**

$$\mathscr{L}\{f(t)\} = \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt,$$

let $t = u + T$, then the 2nd term becomes

$$\int_T^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-s(u+T)} f(u+T)\, du$$

$$= e^{-sT} \int_0^\infty e^{-su} f(u)\, du = e^{-sT} \mathscr{L}\left\{f(t)\right\}$$

Therefore

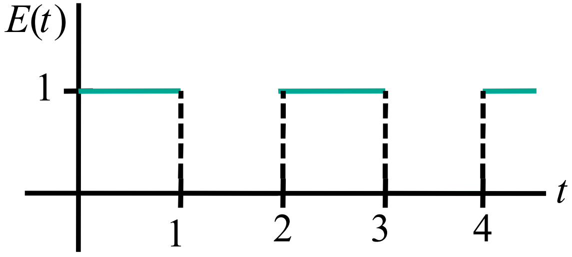
$$\mathscr{L}\left\{f(t)\right\} = \int_0^T e^{-st} f(t)\, dt + e^{-sT} \mathscr{L}\left\{f(t)\right\}$$

$$\to \mathscr{L}\left\{f(t)\right\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$

Example: Square-Wave Transform

- Find the transform of a square-wave.
  Solution:
  One period of $E(t)$ can be defined as:

$$E(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$$



$$\mathscr{L}\{E(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} E(t)\, dt$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} \cdot 1\, dt + \int_1^2 e^{-st} \cdot 0\, dt \right]$$

$$= \frac{1}{1-e^{-2s}} \cdot \frac{1-e^{-s}}{s} = \frac{1}{s(1+e^{-s})}.$$

Example: Periodic Input Voltage (1/3)

- The DE for $i(t)$ in a single-loop LR series circuit is

$$L\frac{di}{dt}+Ri=E(t).$$

Determine $i(t)$ when $i(0)=0$ and $E(t)$ is the square-wave as in the previous example.

Solution:

$$LsI(s)+RI(s)=\frac{1}{s(1+e^{-s})}\;\rightarrow\;I(s)=\frac{1/L}{s(s+R/L)}\cdot\frac{1}{(1+e^{-s})}$$

Since

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots\;\rightarrow\;\frac{1}{1+e^{-s}}=1-e^{-s}+e^{-2s}-e^{-3s}+\ldots$$

Example: Periodic Input Voltage (2/3)

Since $$\frac{1}{s(s+R/L)}=\frac{L/R}{s}-\frac{L/R}{s+R/L},$$

we have

$$I(s)=\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+R/L}\right)(1-e^{-s}+e^{-2s}-e^{-3s}+\cdots).$$

By applying the *t*-axis translation theorem:

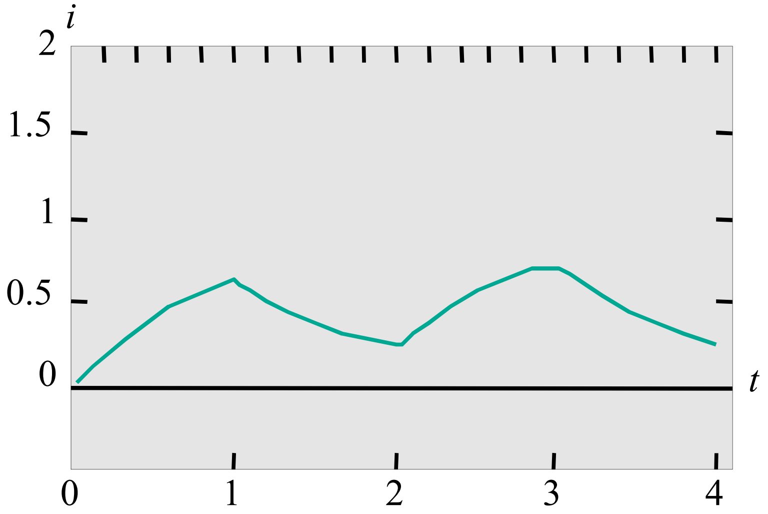
$$i(t)=\frac{1}{R}\left(1-u(t-1)+u(t-2)-u(t-3)+\cdots\right)$$
$$-\frac{1}{R}\left(e^{-Rt/L}-e^{-R(t-1)/L}u(t-1)+e^{-R(t-2)/L}u(t-2)-\cdots\right)$$

Example: Periodic Input Voltage (3/3)

Therefore

$$i(t) = \frac{1}{R}(1 - e^{-Rt/L}) + \frac{1}{R}\sum_{n=1}^{\infty}(-1)^n(1 - e^{-R(t-n)/L})u(t-n).$$

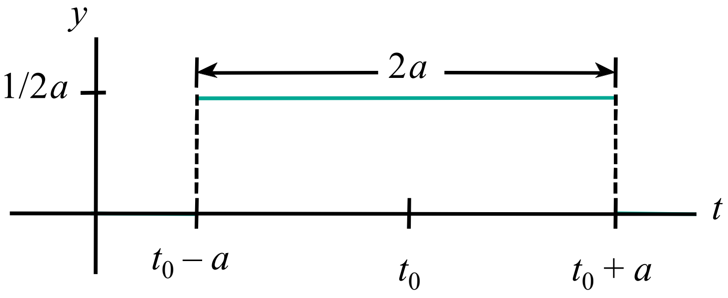
For example, if $R = 1$, $L = 1$, and $0 \le t < 4$, we have



Unit Impulse

- Quite often, the input to a physical system is a short period, large magnitude function. This type of function can be described by

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \dfrac{1}{2a}, & t_0 - a \le t < t_0 + a \\ 0, & t \ge t_0 + a \end{cases}.$$



The function $\delta_a(t - t_0)$ is called unit impulse because

$$\int_0^\infty \delta_a(t-t_0)\, dt = 1.$$

Dirac Delta Function

- ☐ Define $\delta(t-t_0)=\lim_{a\to 0}\delta_a(t-t_0)$. The function $\delta(t-t_0)$ is called Dirac delta function. $\delta(t-t_0)$ is characterized by:

$$(i)\ \delta(t-t_0)=\begin{cases}\infty, & t=t_0\\ 0, & t\neq t_0\end{cases},$$

$$(ii)\ \int_0^\infty \delta(t-t_0)\,dt=1.$$

Transform of $\delta(t-t_0)$

- **Theorem:** For $t > 0$, $\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$.

  Proof:

$$\delta_a(t-t_0) = \frac{1}{2a}\left[u(t-(t_0-a))-u(t-(t_0+a))\right],$$

$$\mathscr{L}\left\{\delta_a(t-t_0)\right\} = \frac{1}{2a}\left[\frac{e^{-s(t_0-a)}}{s}-\frac{e^{-s(t_0+a)}}{s}\right] = e^{-st_0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right),$$

$$\mathscr{L}\left\{\delta(t-t_0)\right\} = \lim_{a\to 0}\mathscr{L}\left\{\delta_a(t-t_0)\right\} = e^{-st_0}\lim_{a\to 0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right) = e^{-st_0}.$$

Note that $\mathscr{L}\{\delta(t)\} = 1$. $\delta(t)$ is not a “normal” function since $\mathscr{L}\{\delta(t)\} \to 1$ as $s \to \infty$.

Example: Two IVPs (1/2)

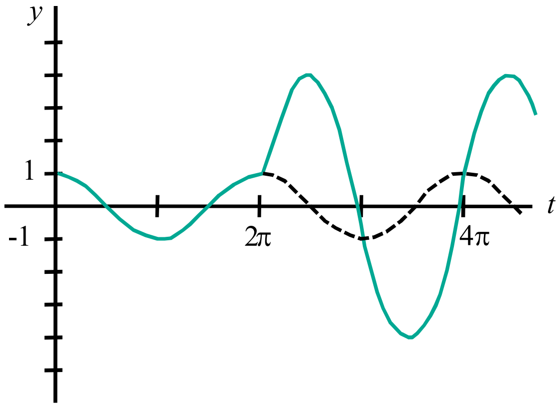
- ☐ Solve $y'' + y = 4\delta(t - 2\pi)$, with initial conditions (a) $y(0) = 1, y'(0) = 0$, and (b) $y(0) = 0, y'(0) = 0$.

  Solution (a):
  The Laplace transform is: $s^2Y(s) - s + Y(s) = 4e^{-2\pi s}$,

$$Y(s) = \frac{s}{s^2+1} + \frac{4e^{-2\pi s}}{s^2+1}.$$

$$\to y(t) = \cos t + 4\sin(t - 2\pi)u(t - 2\pi).$$

$$\to y(t) = \begin{cases} \cos t, & 0 \le t < 2\pi \\ \cos t + 4\sin t, & t \ge 2\pi \end{cases}.$$



Example: Two IVPs (2/2)

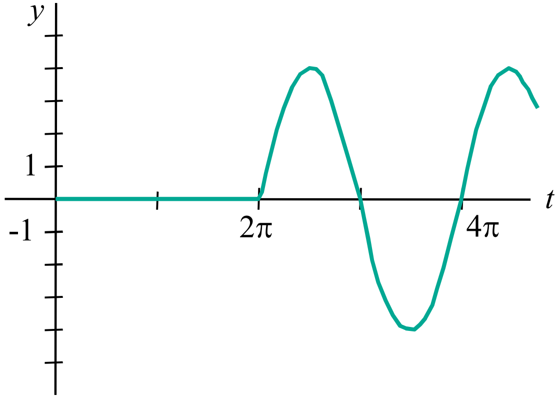
- ☐ Solution (b)

  The Laplace transform is $Y(s) = \dfrac{4e^{-2\pi s}}{s^2+1}$.

  Therefore,

$$y(t) = 4\sin(t-2\pi)u(t-2\pi)$$

$$= \begin{cases} 0, & 0 \le t < 2\pi \\ 4\sin t, & t \ge 2\pi \end{cases}.$$



Impulse Response

- Consider a 2nd-order linear system with unit impulse input at $t = 0$:

$$a_2 x'' + a_1 x' + a_0 x = \delta(t), \quad x(0) = 0,\ x'(0) = 0.$$

Applying Laplace transform to the system:

$$X(s) = \frac{1}{a_2 s^2 + a_1 s + a_0} = \frac{1}{Z(s)} = W(s) \ \rightarrow\ x(t) = \mathscr{L}^{-1}\left\{\frac{1}{Z(s)}\right\} = w(t).$$

$w(t)$ is the zero-state response of the system to a unit impulse, therefore, $w(t)$ is called the impulse response of the system.

Linear Dynamic Systems

- Recall that for a general linear dynamic system, we have

$$X(s)=\frac{F(s)}{Z(s)}+\frac{I(s)}{Z(s)}.$$

$W(s)=1/Z(s)$ is called the transfer function of the system. Note that

$$x(t)=\underbrace{\mathscr{L}^{-1}\{W(s)F(s)\}}_{\text{zero-state response}}+\underbrace{\mathscr{L}^{-1}\{W(s)I(s)\}}_{\text{zero-input response}}.$$