First-Order Differential Equations

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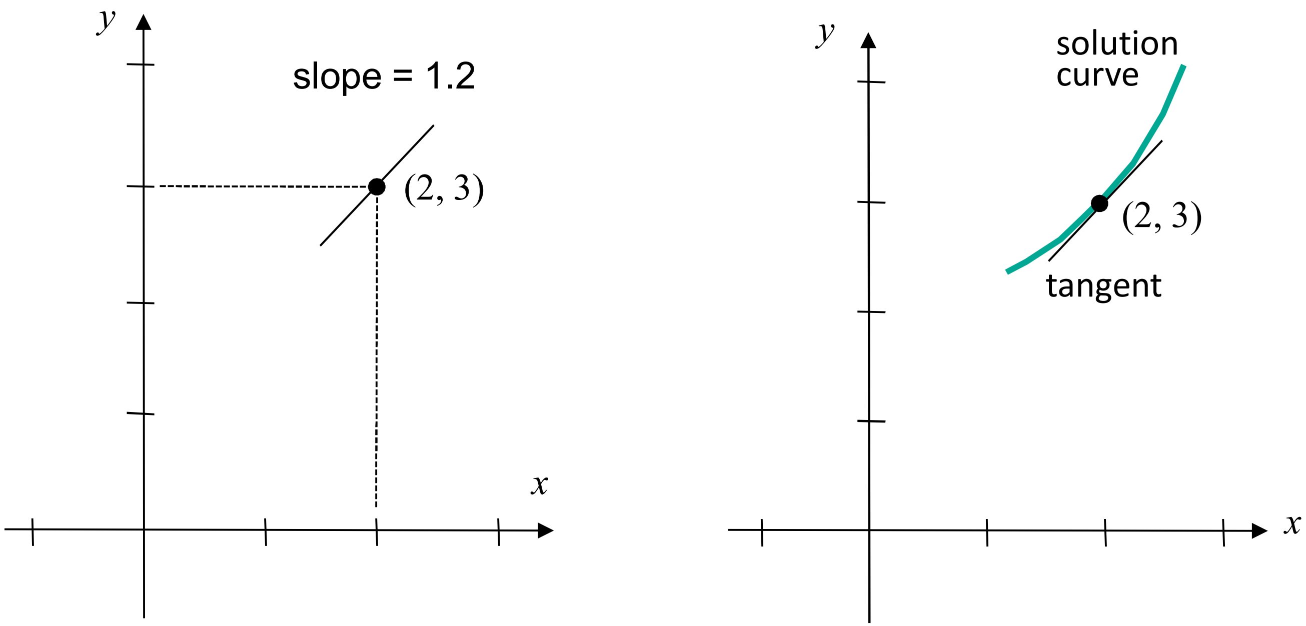
9/16/2019

Solution Curves without Solving DE

- Sometimes, just by looking at the differential equation, we can learn useful information about its solutions:
  - The solution curve $y = y(x)$ of a first order DE $dy/dx = f(x, y)$ on its interval of definition I must possess a tangent line at each point $(x, y(x))$, and must have no breaks.
  - The slope of the tangent line at $(x, y(x))$ on a solution curve is the value of the first derivative dy/dx at this point.
  - A (very small) line segment at $(x, y(x))$ that has the slope $f(x, y)$ is called **lineal element** of the solution curve.

Example of Lineal Element

- Consider $dy/dx = f(x, y) = 0.2xy$, the slope of the lineal element of the solution curve at $(2, 3)$ is $f(2, 3) = 1.2$.

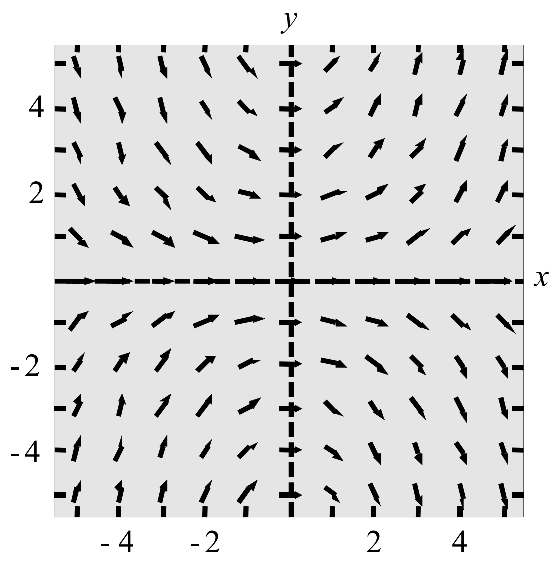


Lineal element at (2, 3)

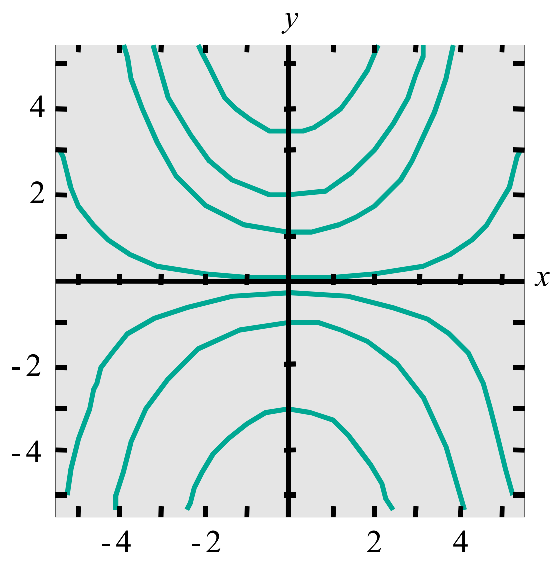
Lineal element is tangent to solution curve passes (2,3)

Direction Field

- The collection of the lineal elements on a rectangular grid on the xy-plane is called a direction field or a slope field of the DE $dy/dx = f(x, y)$.
- A single solution curve on the x–y plane will follow the flow pattern of the slope field.



The slope field of $dy/dx = 0.2xy$.



Solution family: $y = ce^{0.1x^2}$

Increasing/Decreasing of a Solution

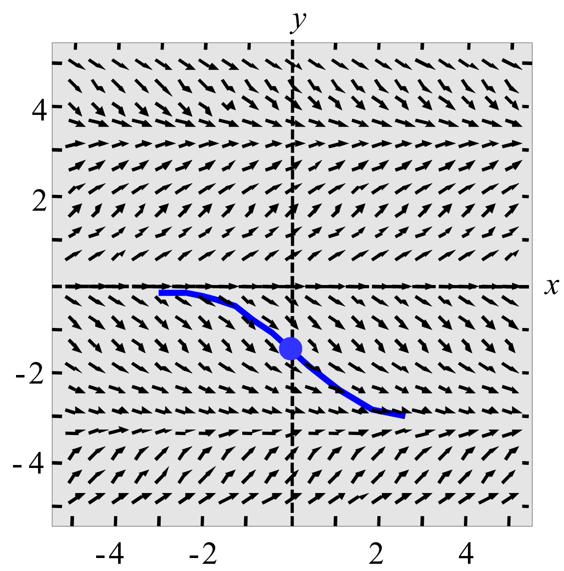
- If $dy/dx > 0$ for all x on the interval of definition I, then the differentiable function $y(x)$ is increasing on I.

- If $dy/dx < 0$ for all x on the interval of definition I, then the differentiable function $y(x)$ is decreasing on I.

Example: Approximating a Solution

- We can use a slope field to approximate the IVP, $dy/dx = \sin y$, $y(0) = -3/2$:
  1) Define the direction field around $y = 0$
  2) Constraint 1: the solution must pass $(0, -3/2)$
  3) Constraint 2: the slope of the solution curve must be 0 when $y = 0$ and $y = -\pi$

→ the solution curve can be approximated as in the figure.



Autonomous First-Order DEs

- A DE in which the independent variable does not appear explicitly is said to be **autonomous**.

- If x is the independent variable, an autonomous DE can be written as $F(y, y') = 0$, or $dy/dx = f(y)$.

- Example: If $y(t)$ is a function of time, then the following DE is autonomous and **time-independent**:

$$\frac{dy}{dt} = 1 + y^2$$

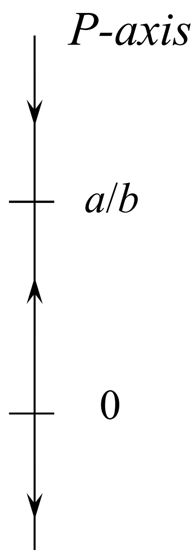
Critical Points

- In $dy/dx = f(y)$, if $f(c) = 0$, then c is called the **critical point** of the autonomous DE. A critical point is also refer to as an equilibrium point or a stationary point.

- If c is a critical point of $dy/dx = f(y)$, then $y(x) = c$ is a constant solution of the autonomous equation. This is also called an **equilibrium solution**.

Example: Autonomous DE

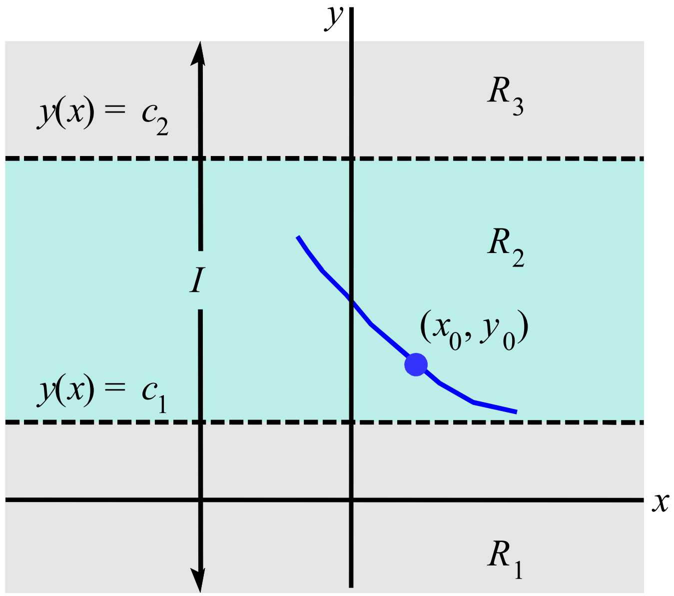
- The DE, $dP/dt = (a - bP)P$, $a > 0$, $b > 0$, is autonomous. Let $(a - bP)P = 0$, we have two critical points: 0 and a/b.
- The sign of $f(P) = P(a - bP)$ can be shown in a phase portrait



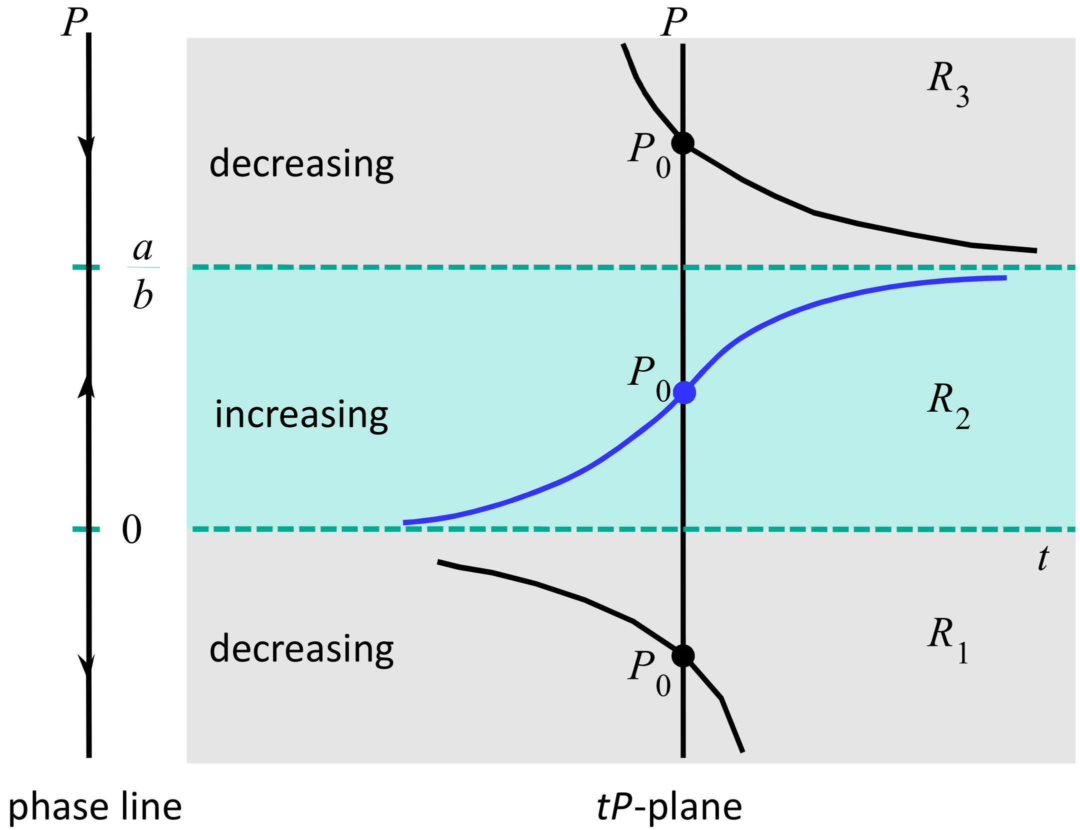
Interval	Sign of $f(P)$	$P(t)$	Arrow
$(-\infty, 0)$	minus	decreasing	down
$(0, a/b)$	plus	Increasing	Up
$(a/b, \infty)$	minus	decreasing	down

Solution Curve Properties

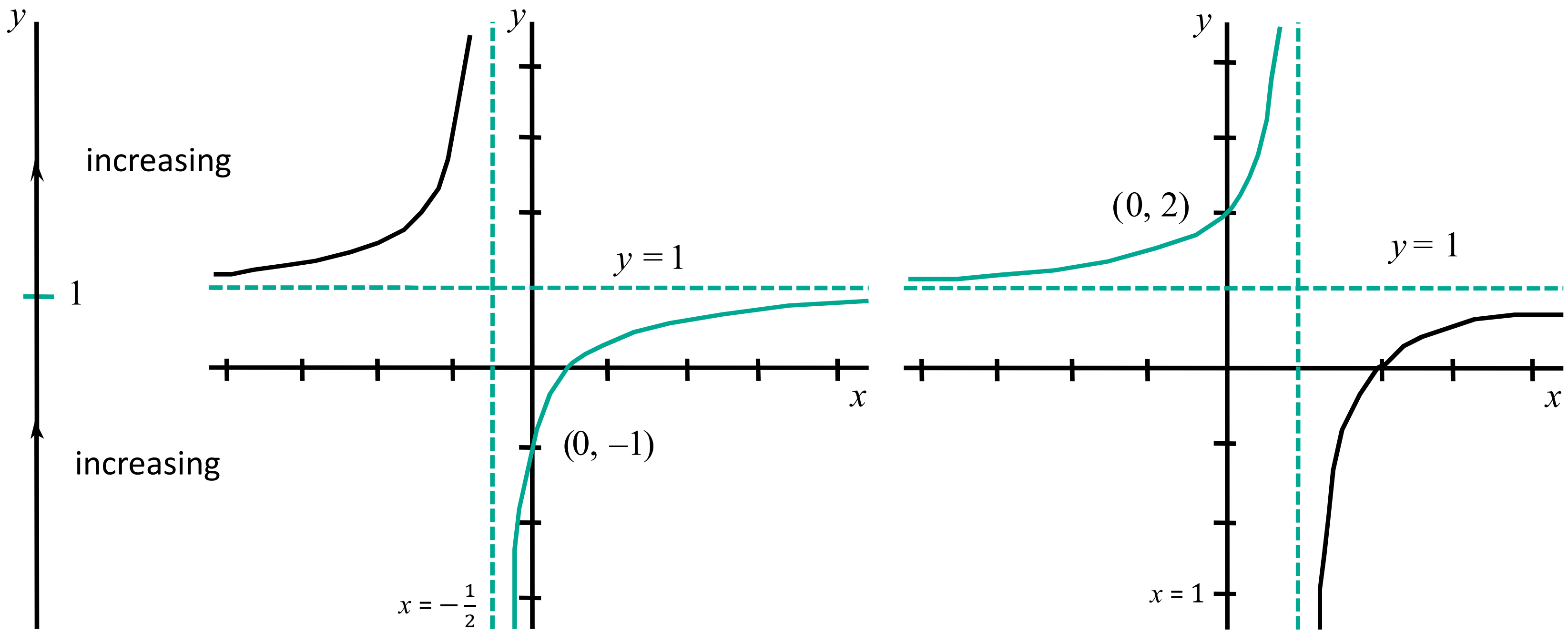
- The solution space can be divided into several regions by equilibrium solutions:
  - $y(x)$ is bounded
  - $f(y) > 0$ or $f(y) < 0$ for all x in a sub region
  - $y(x)$ is strictly monotonic
  - If $y(x)$ is bounded above or below by a critical point, $y(x)$ approaches this point either as $x \to \infty$ or as $x \to -\infty$.



Example: $dP/dt = P(a - bP)$ Revisited



Example: $dy/dx = (y - 1)^2$



(a) Phase line (b) xy-plane, $y(0) < 1$ (c) xy-plane, $y(0) > 1$

Attractors and Repellers

- The solution curve of a first order DE near a critical point c exhibits one of the following three behaviors:
  - Solution curves approach c from either sides. c is called asymptotically stable or an attractor.
  - All solution curves starts near c move away from c. c is called unstable critical point or a repeller.
  - Solution curves approach c from one side and move away from c from the other side. c is called semistable.

Solution by Integration

- If the DE can be expressed in normal form, $f(x, y) = g(x)$, the equation can be solved by integration.

  Since,

  $$\frac{dy}{dx} = g(x)$$

  Integrating both sides, we have:

  $$y = \int g(x)dx = G(x) + c$$

  where $G(x)$ is the indefinite integral of $g(x)$.

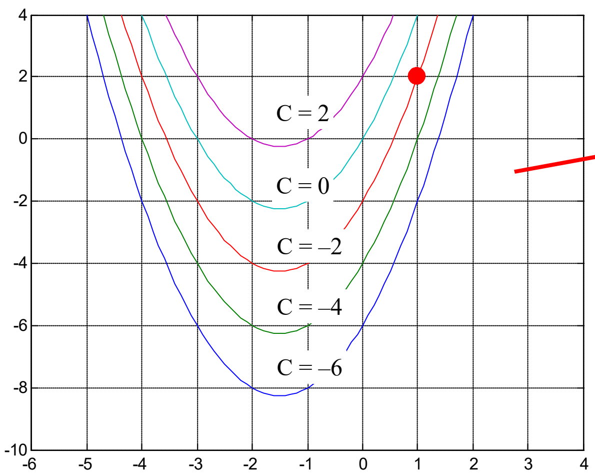
Example: Solution by Integration

- Solving the initial value problem

$$\frac{dy}{dx} = 2x + 3, \quad y(1) = 2.$$

By integrating both sides, we have

$$y(x) = \int (2x+3)dx = x^2 + 3x + c.$$



Family of solution curves

Solution that passes through the initial condition $(1, 2)$ is the curve with $C = -2$

2^{nd}-Order Solution by Integration

- If we have a second-order DE of the special form:

$$\frac{d^2y}{dx^2} = g(x),$$

we have

y'

$$\int y''(x)dx = \int g(x)dx = G(x) + C_1,$$

where G is an anti-derivative of g and C_1 is an arbitrary constant. Therefore,

$$y(x) = \int y'(x)dx = \int [G(x) + C_1]dx = \int G(x)dx + C_1 x + C_2,$$

where C_2 is a second arbitrary constant.

Separable Equations (1/2)

- A first order DE of the form

$$\frac{dy}{dx} = g(x)h(y) = g(x)/f(y), \quad \text{where } f(y) = 1/h(y)$$

is said to be separable or to have separable variables. Divide both side by $h(y)$, the DE becomes

$$f(y)\frac{dy}{dx} = g(x)$$

Integrating both sides w.r.t. x, we have

$$\int f(y(x))\frac{dy}{dx}dx = \int g(x)dx + C.$$

Separable Equations (2/2)

Cancelling the differential term dx, we have

$$\int f(y)dy = \int g(x)dx + C.$$

If the two anti-derivatives

$$F(y) = \int f(y)dy \quad \text{and} \quad G(x) = \int g(x)dx$$

can be found, we have the family of equations

$$F(y(x)) = G(x) + C$$

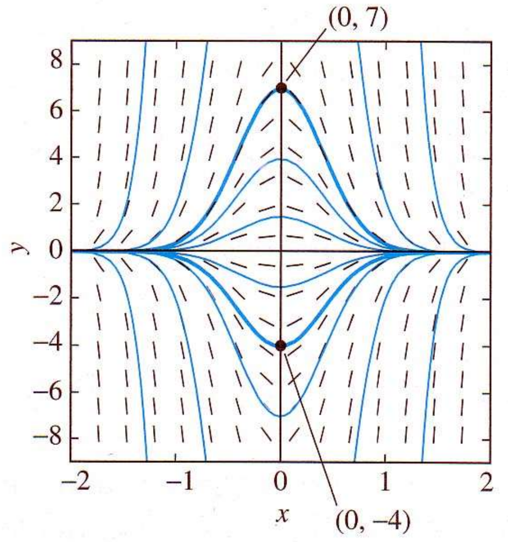
that conforms to the differential equation.

Example: $dy/dx = -6xy,\ y(0) = 7$

- Rearranging the equation, we have $dy/y = -6xdx$, therefore,

$$\int dy/y = \int -6xdx$$

$$\rightarrow \ln|y| = -3x^2 + C_1.$$



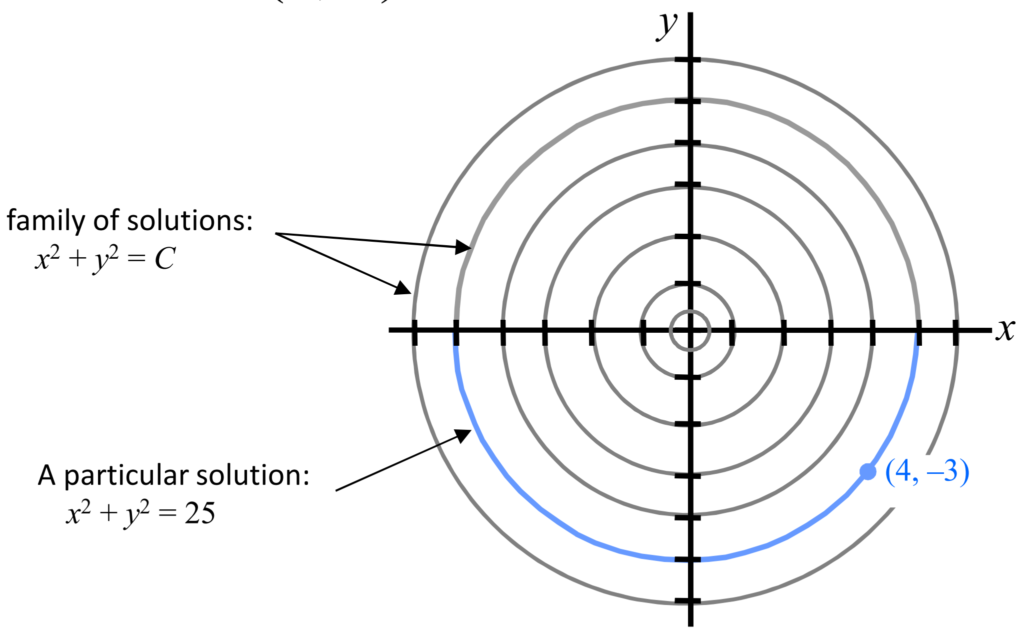
Thus $|y| = e^{-3x^2} e^{C_1}$ or $y = \pm e^{C_1} e^{-3x^2}$.
We have, $y = C_2 e^{-3x^2}$, $C_2 = \pm e^{C_1} \in R$.
However, $y = 0$ is also a solution.
Note that as $C_2 \rightarrow 0$, $y \rightarrow 0$.

Since $y(0) = 7$, the particular solution is $y = 7e^{-3x^2}$.

Example: $dy/dx = -x/y$, $y(4) = -3$

- Since $\int y dy = -\int x dx$, we have $y^2/2 = -x^2/2 + c_1$.

  The solution must pass $(4, -3)$, thus, $c_1 = 25/2$.
  $\rightarrow$ the solution is the lower half-circle of radius 5 centered at $(0, 0)$.



Losing a Solution

- Some care should be exercised when separating variables, since the variable divisors could be zero in some cases.
- If r is a zero of $h(y)$, then $y = r$ is a constant solution of the DE. However, $y = r$ may not show up in the family of solutions. Recall that this is called a singular solution.

Example: $dy/dx = y^2 - 4$

- Since $y^2 - 4$ is separable

$$\int \frac{dy}{y^2-4} = \int dx \longrightarrow \int \left[\frac{1/4}{y-2} - \frac{1/4}{y+2} \right] dy = \int dx$$

$$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x + c_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + c_2, \quad \text{or} \quad \frac{y-2}{y+2} = ce^{4x}, c = \pm e^{c_2}$$

$$\longrightarrow \quad y = 2\frac{1+ce^{4x}}{1-ce^{4x}}$$

Note: The solutions $y = \pm 2$ have been excluded in the first step!

Example: $(e^{2y} - y)\cos x\dfrac{dy}{dx} = e^{y}\sin 2x,\ y(0) = 0$

- Solve the IVP by dividing both sides by $e^{y}\cos x$, then multiply both sides by dx, we have

$$\frac{e^{2y} - y}{e^{y}}dy = \frac{\sin 2x}{\cos x}dx$$

$$\int\left(e^{y} - ye^{-y}\right)dy = 2\int \sin x\, dx$$

$$e^{y} + ye^{-y} + e^{-y} = -2\cos x + c,\quad y(0) = 0$$

$$\longrightarrow \quad c = 4$$

Linear First Order DE

- A first-order differential equation of the form:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

  is said to be a **linear equation**. When $g(x) = 0$, the linear equation is said to be **homogeneous**, otherwise it is non-homogeneous.
- Dividing both side of (1) by the leading coefficient $a_1(x)$, we have the **standard form**:

$$\frac{dy}{dx} + P(x)y = f(x)$$

Solving the 1st-Order Standard Form

- The solution of $dy/dx + P(x)y = f(x)$ can be derived by multiplying both sides of the equation by a special function $\mu(x)$. We want the function $\mu(x)$ to satisfy the property:

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y \equiv \mu\left(\frac{dy}{dx} + P(x)y\right) = \mu f(x).$$

Thus $d\mu/dx = \mu P(x) \;\to\; \mu = e^{\int P(x)dx}$.

The function $\mu(x)$ is called the integrating factor.

Solution by Integrating Factors

- We can solve the DE by multiplying both sides of the standard form by $e^{\int P(x)dx}$, thus:

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = f(x)e^{\int P(x)dx}$$

$$\longrightarrow \quad \frac{d}{dx}\left[ye^{\int P(x)dx}\right] = f(x)e^{\int P(x)dx}$$

$$\longrightarrow \quad ye^{\int P(x)dx} = \int f(x)e^{\int P(x)dx}dx + c$$

Dropping Integrating Factor Constant

- Note that you do not need to keep the constant when computing the anti-derivative of the integrating factor. Assume that $G(x)$ is the anti-derivative of $P(x)$, since

$$e^{\int P(x)dx} = e^{G(x) + c} = c_1 e^{G(x)},$$

  The constant $c_1 = e^c$ will simply be cancelled out on both side of the differential equation.

Example: Solve $dy/dx - 3y = 6$

- Solution:

$$e^{\int(-3)\,dx} = e^{-3x}$$

$$\longrightarrow \quad e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}$$

$$\longrightarrow \quad \frac{d}{dx}\left[e^{-3x}y\right] = 6e^{-3x}$$

$$\longrightarrow \quad e^{-3x}y = -2e^{-3x} + c$$

$$\longrightarrow \quad y = -2 + ce^{3x}, \quad -\infty < x < \infty$$

General Solution on I

- If $P(x)$ and $f(x)$ in the standard form are continuous on an open interval I, then

$$y = ce^{-\int P(x)dx} + e^{-\int P(x)dx}\int e^{\int P(x)dx} f(x)dx$$

is a general solution of $dy/dx + P(x)y = f(x)$.

That is, every solutions on I has the form. In other words, there is no singular solution for the linear 1st order differential equation on I.

Particular Solution on I

- Given an initial condition $y(x_0) = y_0$ to the linear first order DE $dy/dx + P(x)y = Q(x)$ on I where $P(x)$ and $Q(x)$ are continuous, the particular solution of the DE has the form:

$$y(x) = e^{-\int_{x_0}^{x} P(t)dt} \left[y_0 + \int_{x_0}^{x} e^{\int_{x_0}^{t} P(u)du} Q(t)dt \right]$$

Note that it is easy to verify that $y(x_0) = y_0$.

Example: $(x^2-9)dy/dx + xy = 0$

- Solution:

$$\frac{dy}{dx} + \frac{x}{x^2-9}y = 0, \quad \therefore P(x) = \frac{x}{(x^2-9)}$$

$P(x)$ is continuous on $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$.
Thus, the integrating factor is:

$$e^{\int \frac{x}{(x^2-9)}dx} = e^{1/2\ln\left|x^2-9\right|} = \sqrt{\left|x^2-9\right|}, \; x \neq -3, 3.$$

Therefore, $\frac{d}{dx}\left[\sqrt{\left|x^2-9\right|}y\right] = 0.$

$$\longrightarrow \quad \sqrt{\left|x^2-9\right|}y = c, \;\; x \neq -3,3.$$

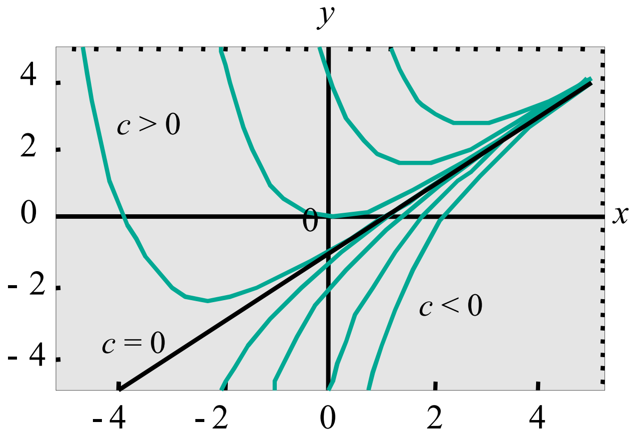
Example: IVP $y' + y = x$, $y(0) = 4$

- Since $P(x) = 1$ and $Q(x) = x$ are continuous on $(-\infty, \infty)$, we have integrating factor $e^{\int dx} = e^x$:

$$\frac{d}{dx}\left[e^x y\right] = xe^x$$

transient term

$$\longrightarrow \quad y = (x-1) + ce^{-x}, \quad -\infty < x < \infty$$



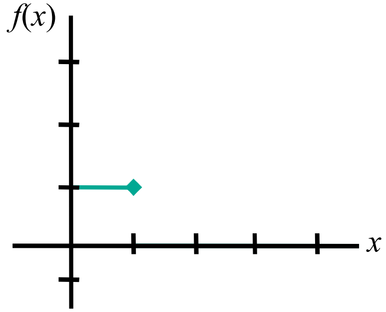
Example: Discontinuous $f(x)$

- Find a continuous function satisfying

$$\frac{dy}{dx} + y = f(x),\ f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases} \text{ and } y(0) = 0.$$

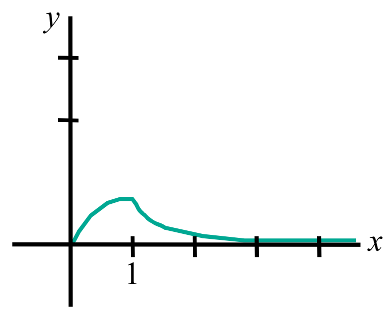
Solution:

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1 \\ c_2 e^{-x}, & x > 1 \end{cases}$$



$\rightarrow$ find c_2 so that

$$\lim_{x \to 1^+} y(x) = y(1)$$



Non-elementary Functions

- Some simple function do not possess antiderivatives that are elementary functions, and integrals of this kind of functions are called **non-elementary**.
- The integrations of non-elementary functions can only be solved by numerical methods.

Example:

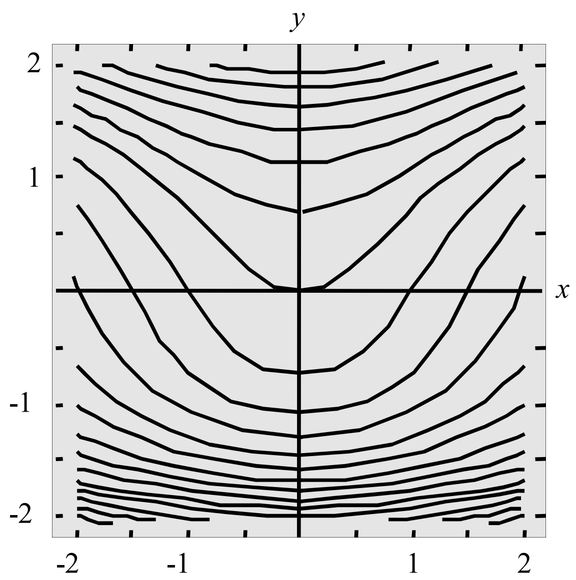
$$\int e^{-x^2} dx$$

$$\int \sin x^2 dx$$

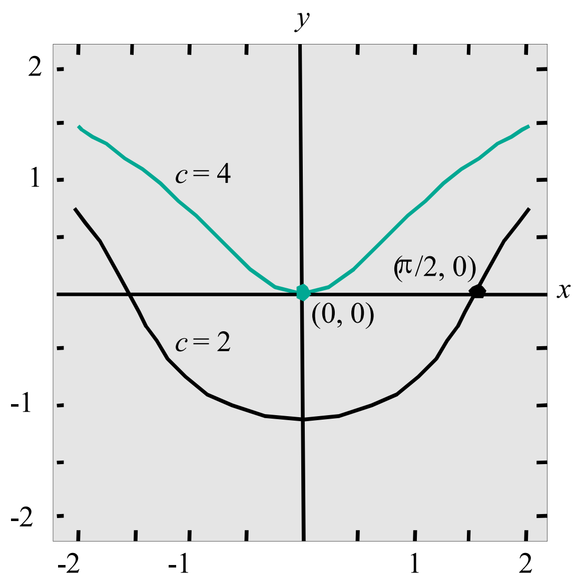
$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Level Curves and Family of Solutions

- In multivariate calculus, for a function of two variables, $z = G(x, y)$, the curves defined by $G(x, y) = c$ (c is a constant) are called level curves of the function.



Level curves of $e^y + ye^{-y} + e^{-y} + 2\cos = c$



Solutions of IVPs

Differentials of Two-Variable Functions

- If $z = f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its total differential is:

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

If $f(x, y) = c$, we have:

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0.$$

$\rightarrow$ Given a one-parameter family of curves $f(x, y) = c$, we can derive a first order DE.

Example:

- If $x^2 - 5xy + y^3 = c$, then taking the differential gives

$$(2x - 5y)\, dx + (-5x + 3y^2)\, dy = 0.$$

Question: can we think reversely?

Exact Equations

- A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region R of the xy-plane if it is the total differential of some function $f(x, y)$.

- A first-order differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential.

Criterion for an Exact Differential

- **Theorem**: Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$.

  Then a necessary and sufficient condition that $M(x, y)\, dx + N(x, y)\, dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Proof of the Necessity

- If $M(x, y)\,dx + N(x, y)\,dy$ is exact, there exists some function f such that for all x in R,

$$M(x, y)dx + N(x, y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Therefore, $M(x, y) = \partial f/\partial x$, and $N(x, y) = \partial f/\partial y$, and

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial x}.$$

#

Proof of the Sufficiency (1/2)

- Note that we have

$$\frac{\partial f}{\partial x} = M(x, y) \;\rightarrow\; f(x, y) = \int M(x, y)dx + g(y),$$

where $g(y)$, shall be a function of y. Since we want

$$\frac{\partial f}{\partial y} = N(x, y) \;\rightarrow\; N(x, y) = \frac{\partial}{\partial y}\int M(x, y)dx + g'(y),$$

therefore, $$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)dx$$

If we can prove that $g'(y)$ is a function of y alone, integrating $g'(y)$ w.r.t. y, gives us the solution.

Proof of the Sufficiency (2/2)

- Since $$\frac{\partial}{\partial x}\left(N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx\right)=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}$$

  and $\partial M/\partial y = \partial N/\partial x$, we have $\partial/\partial x[g'(y)] = 0$.
  Thus, $g'(y)$ is a function of y alone.

  In this case, the solution is

$$f(x,y)=\int M(x,y)dx+\int\left(N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx\right)dy.$$

#

Observations

- The solution to an exact eq. $M(x, y)\, dx + N(x, y)\, dy = 0$ is

$$f(x,y) = \int M(x,y)dx + \int \left(N(x,y) - \frac{\partial}{\partial y} \int M(x,y)dx \right) dy = c,$$

where c is a constant parameter.

- The method of solution can start from $\partial f/\partial y = N(x, y)$ as well. Then, we have

$$f(x,y) = \int N(x,y)dy + h(x)$$

$$h'(x) = M(x,y) - \frac{\partial}{\partial x} \int N(x,y)dy$$

Example: $2xy\,dx + (x^2 - 1)dy = 0$

- Solution:
  Since $M(x, y) = 2xy$, $N(x, y) = x^2 - 1$, we have $\partial M/\partial y = 2x = \partial N/\partial x$ so the equation is exact, and there exists $f(x, y)$ such that $\partial f/\partial x = 2xy$ and $\partial f/\partial y = x^2 - 1$.

  Integrating the first equation$\rightarrow f(x, y) = x^2y + g(y)$
  Take the partial derivative of y, equate it with $N(x, y)$, we have $x^2 + g'(y) = x^2 - 1$. Therefore, $g'(y) = -1$, and $f(x, y) = x^2y - y$. The implicit solution is $x^2y - y = c$.

Example: An IVP of Exact Equation

- Solve

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)},\ y(0) = 2.$$

Solution:

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}$$

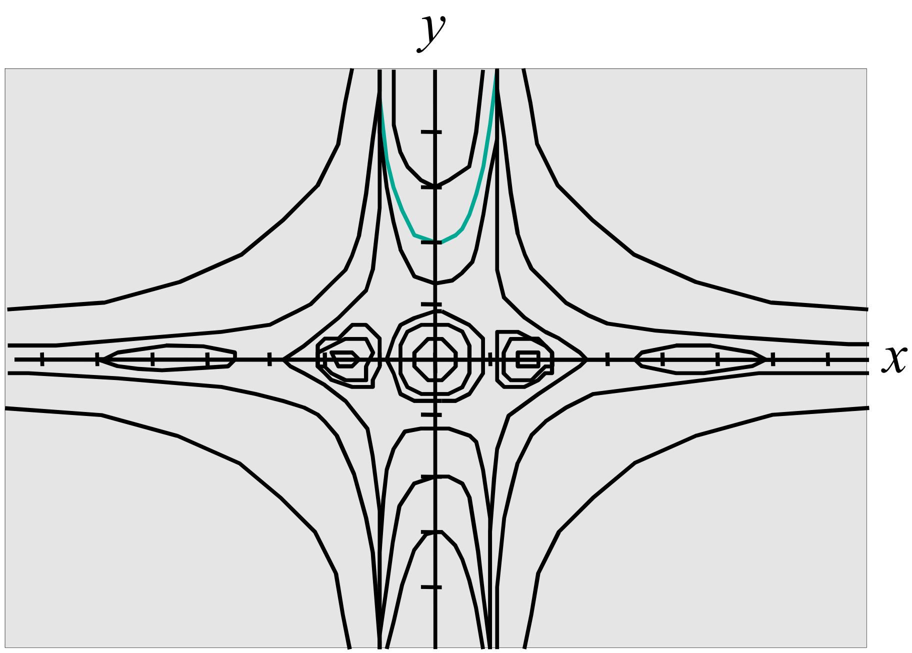
$$\rightarrow \frac{\partial f}{\partial y} = y(1-x^2),\ \ f(x,y) = \frac{y^2}{2}(1-x^2) + h(x)$$

$$\rightarrow \frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2$$

$$\rightarrow h(x) = -\int (\cos x)(-\sin x dx) = -\tfrac{1}{2}\cos^2 x$$

Example: An IVP (cont.)

The implicit solution is $y^2(1-x^2)-\cos^2 x=c$.
Substitute the initial condition $y(0)=2$ into the implicit solution, we have $c=3$.



Integrating Factors for Exactness

- Can we multiply a non-exact equation by an integrating factors $\mu(x, y)$ to make it exact? That is, can we make

$$\mu(x, y)M(x, y)\ dx + \mu(x, y)N(x, y)\ dy = 0$$

an exact differential equation? To achieve this goal, $\mu(x, y)$ must satisfy

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0.$$

- In practice, a proper $\mu(x, y)$ is not easy to find unless it happens to be a function of x or y alone. If $\mu(x, y) = \mu(x)$,

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$ $\rightarrow$ Separable equation if $(M_y - N_x)/N$ contains x alone.

Solution by Substitutions

- We can substitute $dy/dx = f(x, y)$ with $y = g(x, u)$, where u is a function of x, to solve for the solution.

  By chain rule:
  $$\frac{dy}{dx} = g_x(x,u) + g_u(x,u)\frac{du}{dx},$$
  then
  $$f(x, g(x,u)) = g_x(x,u) + g_u(x,u)\frac{du}{dx}.$$

  We can then solve for $du/dx = F(x, u)$.
  If $u = \phi(x)$ is the solution, then $y = g(x, \phi(x))$.

Homogeneous Equations (1/2)

- If $f(tx, ty) = t^{\alpha} f(x, y)$ for some real number α, then f is said to be a homogeneous equation of degree α.

  Example: $f(x, y) = x^3 + y^3$ is a homogeneous equation of degree 3.

- Similarly, a first-order DE in differential form

$$M(x, y)dx + N(x, y)dy = 0$$

  is said to be homogeneous if both M and N are homogeneous function of the same degree.

Homogeneous Equations (2/2)

- The meaning of "homogeneous" here is different from the "homogeneous" in Sec. 2.3.
- If M and N are homogeneous functions of degree α, we have:
  $M(x, y) = x^{\alpha}M(1, u)$ and $N(x, y) = x^{\alpha}N(1, u)$, $u = y/x$
  and
  $M(x, y) = y^{\alpha}M(v, 1)$ and $N(x, y) = y^{\alpha}N(v, 1)$, $v = x/y$
- We can turn a homogeneous equation into a separable first order DE using substitution with either $y = ux$ or $x = vy$.

Example: $(x^2 + y^2)dx + (x^2 - xy)dy = 0$

- Solution:
  M and N are 2nd-order homogeneous equation.
  Let $y = ux$, then $dy = u\,dx + x\,du$.
  After substitution, we have

  $$\rightarrow (x^2 + u^2x^2)dx + (x^2 - ux^2)[u\,dx + x\,du] = 0$$
  $$\rightarrow x^2\,(1+u)dx + x^3\,(1-u)\,du = 0$$

  Therefore
  $$\frac{1-u}{1+u}du + \frac{dx}{x} = 0 \rightarrow \left[-1 + \frac{2}{1+u}\right]du + \frac{dx}{x} = 0$$

  $$-u + 2\ln|1+u| + \ln|x| = \ln|c|$$

Bernoulli's Equation

- The DE

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where n is any real number, is called Bernoulli's equation. Note that for $n = 0$ and $n = 1$, it is linear. For any other n, the substitution $u = y^{1-n}$ reduces any equation of this form to a linear equation.

Example: $x\, dy/dx + y = x^2y^2$

- Solution:

$$\frac{dy}{dx} + \frac{1}{x} y = xy^2,$$

substitute with $y = u^{-1}$ and $dy/dx = -u^{-2} du/dx$.

$\rightarrow \dfrac{du}{dx} - \dfrac{1}{x} u = -x$, the integrating factor on $(0, \infty)$

is $e^{-\int dx/x} = x^{-1}$, we have $\dfrac{d}{dx}\left[x^{-1}u\right] = -1$

$x^{-1}u = -x + c \rightarrow y = 1/(-x^2 + cx)$.

Another Reduction to Separation

- A DE of the form

$$\frac{dy}{dx} = f(Ax + By + C)$$

can always be reduced to an equation with separable variables by means of the substitution $u = Ax + By + C,\ B \neq 0$.

Example: $dy/dx = (-2x + y)^2 - 7,\ y(0) = 0$

- Solution:

  Let $u = -2x + y$, then $du/dx = -2 + dy/dx$.

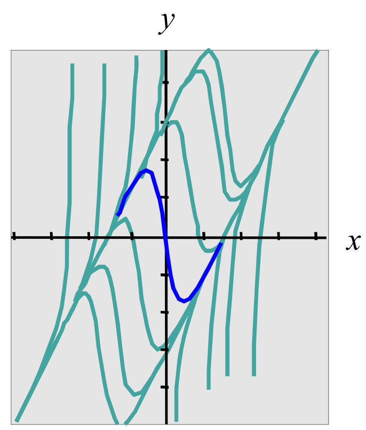
  The DE can be reduced to $du/dx = u^2 - 9$.

$$\rightarrow \frac{du}{(u-3)(u+3)} = dx \rightarrow \frac{1}{6}\left[\frac{1}{u-3} - \frac{1}{u+3}\right]du = dx$$

$$\rightarrow \frac{1}{6}\ln\left|\frac{u-3}{u+3}\right| = x + c_1 \rightarrow \frac{u-3}{u+3} = ce^{6x},\ c = e^{6c_1}$$

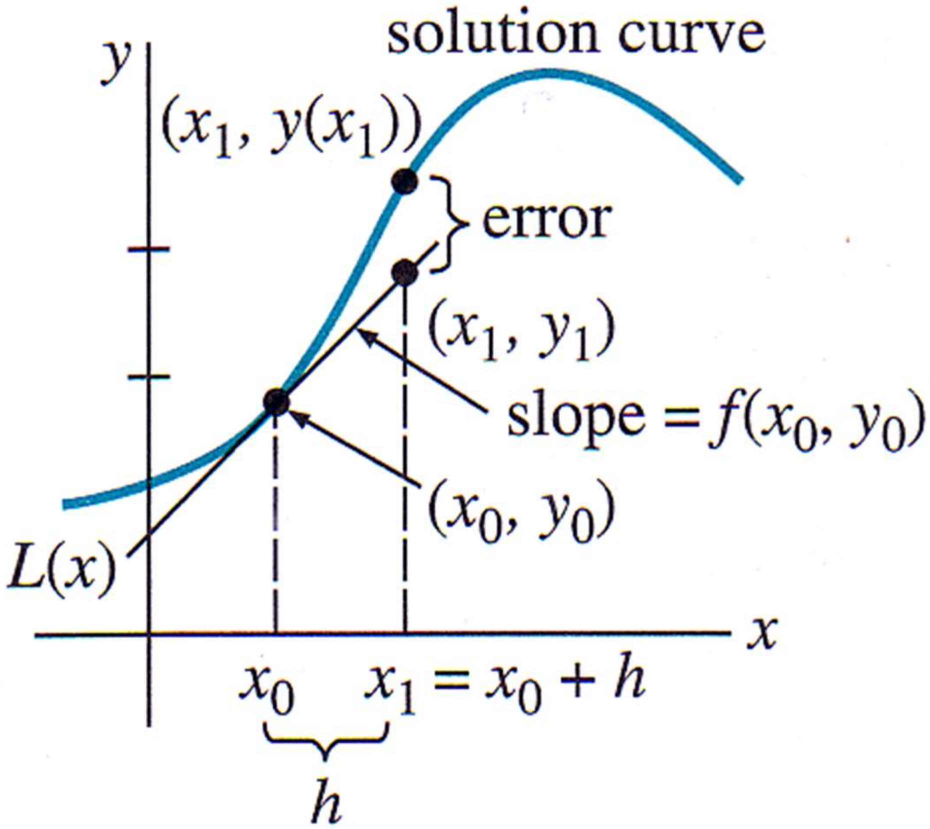
$$\rightarrow y = 2x + \frac{3(1+ce^{6x})}{1-ce^{6x}}$$

$$\rightarrow y(0) = 0, c = -1.$$



Numerical Methods

- The solution of a DE can be approximated using a tangent line:



Euler’s Method

- One can solve the IVP: $y' = f(x, y), y(x_0) = y_0$, numerically using the following procedure:

1. Linearization of $y(x)$ at $x = x_0$: $L(x) = f(x_0, y_0)(x - x_0) + y_0$
2. Replace x in the above equation with $x_1 = x_0 + h$, we have $L(x_1) = f(x_0, y_0)(x_0 + h - x_0) + y_0$ or $y_1 = y_0 + hf(x_0, y_0)$, where $y_1 = L(x_1)$
3. If $h \to 0$ then $y_1 \sim y(x_1)$
4. Use (x_1, y_1) as a new starting point, we have $x_2 = x_1 + h = x_0 + 2h$, and $y(x_2) = y_1 + hf(x_1, y_1)$
5. Recursively, we have $y_{n+1} = y_n + hf(x_n, y_n)$, where $x_n = x_0 + nh$, $n = 0, 1, 2, \ldots$

Error Accumulations

- Numerical solutions are approximations to the exact solution of a DE → approximation errors may become large when x is far away from the initial condition.