Introduction to Differential Equations

National Chiao Tung University
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9/9/2019

Outline of the Course†

- Introduction to differential equations (Chapter 1)
- First-order differential equations (Chapter 2)
- Higher-order differential equations (Chapter 4)
- Modeling with Higher-order differential equations (Chapter 5)
- The Laplace transform (Chapter 7) ← *midterm around this point!*

- Systems of linear 1st-order differential equations (Chapter 8)
- Power series methods (Chapter 6)
- Fourier series methods (Chapter 11)
- Partial differential equations (Chapter 12)

Textbook and Grading Policy

- Textbook:
  - Dennis G. Zill, *Differential Equations with Boundary-Value Problems*, 9th edition, 2018, Cengage Learning.
(高立圖書代理, 顏俊杰 0921-456030)
  - An **alternative** textbook:
Dennis G. Zill, *Differential Equations with Modeling Applications*, 11th edition, 2018, Cengage Learning.
(華泰文化, 蕭瑀倢 0933-838337)
- Grading is based on
  - Pop Quizzes (25%) – from homework assignments
  - Mid-terms exam (35%) – on 11/4/2019
  - Final exam (40%) – on 1/6/2020

Before You Move On …

- Homework #0: Check out the following video:

Raffaello D'Andrea’s TED talks

The astounding athletic power of quadcopters.
Jun 2013

I will be asking you questions on this video in our next class!

Differential Equations

- **Definition:**
  An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation (DE).

- Example:

$$x = f(t) = e^{0.1t^2} \longrightarrow \frac{dx}{dt} = e^{0.1t^2} \cdot (0.2t) \longrightarrow \frac{dx}{dt} = 0.2xt$$

In this course, given a blue equation (behavior of a phenomenon), you want to find out the red equation (the governing rule) behind it

Why Differential Equations

- For dynamic phenomena, we want to predict their long-term behavior by observing and measuring their short-term behavior
  - Long-term behavior of a dynamic system is defined by its underlying rule $\rightarrow$ hard to measure
  - Short-term behavior of a dynamic system is described by its changing characteristics (derivatives) $\rightarrow$ easier to measure

Classification of DE by Type

- Ordinary differential equation (ODE): an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable

$$\frac{dy}{dx} + 5y = e^x$$

- Partial differential equation (PDE): an equation involving the partial derivatives of one or more dependent variables of two or more independent variables

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

Classification of DE by Order

- The order of a differential equation is the order of the highest derivative in the equation.

$$\underset{\text{2nd order}}{\frac{d^2 y}{dx^2}} + 5\left(\underset{\text{1st order}}{\frac{dy}{dx}}\right)^3 - 4y = e^x$$

- An n^{th}-order ODE with one dependent variable can be expressed in the general form:

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

F → a real-valued function of $n+2$ variables

Normal Form of ODE

- $F()$ can be expressed in general in the *normal form*:

$$\frac{d^n y}{dx^n} = f(x, y, y', y'', \ldots, y^{(n-1)})$$

where f is a real-valued function with $n+1$ variables.

For example, the normal forms of the first order and the 2nd-order ODEs are:

$$\frac{dy}{dx} = f(x, y)$$

$$\frac{d^2 y}{dx^2} = f(x, y, y')$$

Classification of DE by Linearity

- An nth-order ODE, F, is said to be linear if F is linear in y, y', …., $y^{(n)}$. That is, F can be expressed as:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

  where $a_i(x)$, $i = 0, \ldots, n$ depend on the independent variable x only

- Example:
  - $(y - x)dx + 4xdy = 0$
  - $y'' - 2y' + y = 0$
  - $\frac{d^3 y}{dx^3} + 3x\frac{dy}{dx} - 5y = e^x$

Nonlinear ODE

- A differential equation with nonlinear functions of the dependent variable or its derivatives.
- Examples: If y is the dependent variable,
  - $(1 - y)y' + 2y = e^x$
  - $d^2y/dx^2 + \sin y = 0$
  - $y^{(4)} + y^2 = 0$

Solution of an ODE

- **Definition:** a solution of an ODE is a function $y(x)$, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an n^{th}-order ODE reduces the equation to an identity.

- That is, a solution $y(x)$ of F satisfies:

$$F(x, y(x), y'(x), y''(x), \ldots, y^{(n)}(x)) = 0, \quad \forall x \in I.$$

- If an ODE has a solution $y(x) = 0$, $\forall x \in I$, then it is called the trivial solution of the ODE.

All Roads Lead to Rome

- If we have a function y:

$$y(x) = Ce^{x^2}, \quad C \in R.$$

Then,

$$\frac{dy}{dx} = C\left(2xe^{x^2}\right) = 2x\left(Ce^{x^2}\right) = 2xy.$$

Thus, it doesn't matter what the constant C is, $y = Ce^{x^2}$ is a solution of the DE $dy/dx = 2xy$.

- Often, a differential equation alone has many solutions; more information is required to resolve ambiguity

Solution is not Guaranteed

- Expressing a phenomenon as a differential equation does not guarantee that it has a solution. Obviously,

$$(y')^2 + y^2 = -1$$

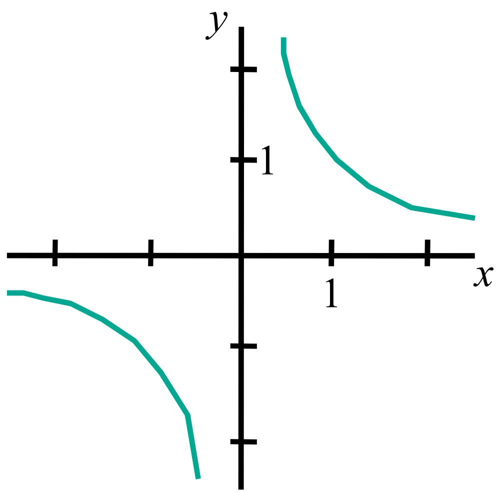
has no (real-valued) solution.

Interval of Definition

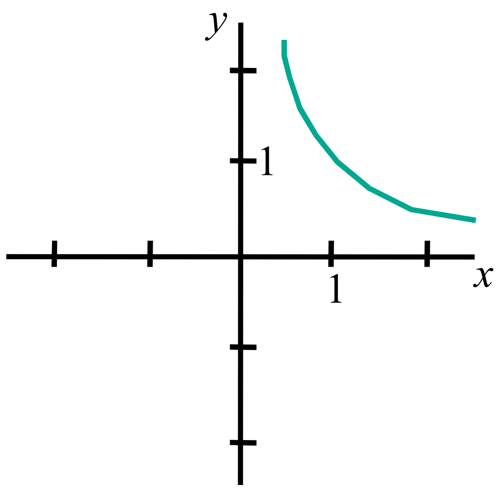
- A solution of an ODE includes a function $y(x)$ and the *interval of definition*, I.
- I is usually referred to as the interval of definition, the interval of existence, the interval of validity, or the domain of the solution.
- I can be an open interval (a, b), a closed interval $[a, b]$, an infinite interval (a, ∞), and so on.

Solution Curve

- The graph of a solution $y(x)$ of an ODE is called a solution curve. Since $y(x)$ is a differentiable function, it is continuous on its interval of definition.
- There maybe a difference between the graph of $y(x)$ and the graph of the solution of the ODE.



$y = 1/x,\ x \neq 0$



$y = 1/x,\ (0, \infty)$

Explicit and Implicit Solutions

- **Definition:** A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is called an explicit solution.

- **Definition:** An equation $G(x, y) = 0$ is said to be an implicit solution of an ODE on an interval I provided that there exists at least one function y that satisfies the relation as well as the differential equation on I.

Verification of an Implicit Solution

- Example:
  The relation $x^2 + y^2 = 25$ is the implicit solution of the differential equation $dy/dx = -x/y$ on the interval $-5 < x < 5$

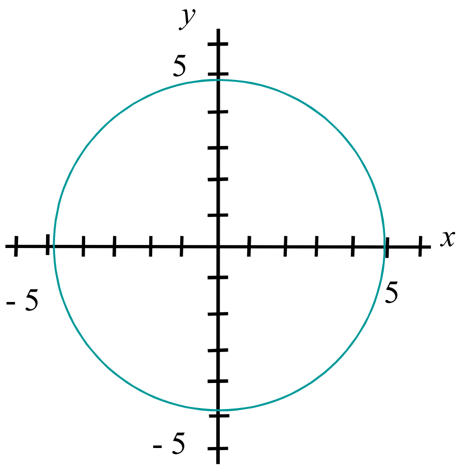
  Verification:

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25) \longrightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\longrightarrow \frac{dy}{dx} = -x / y$$

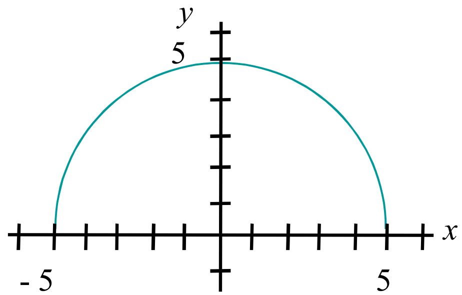
Solving for Explicit Solution

- One can solve an implicit solution for explicit solutions. In the previous example,



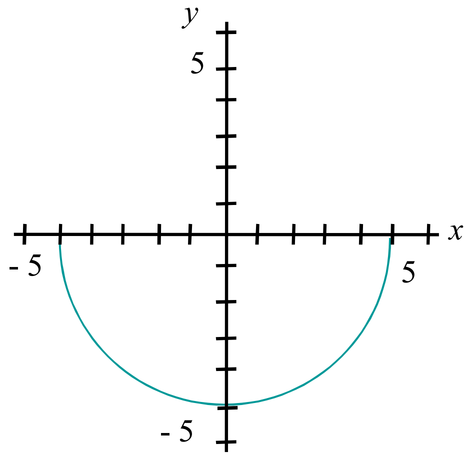
Implicit solution

$x^2 + y^2 = 25$



Explicit solution 1

$y_1 = \sqrt{25 - x^2}, -5 < x < 5$



Explicit solution 2

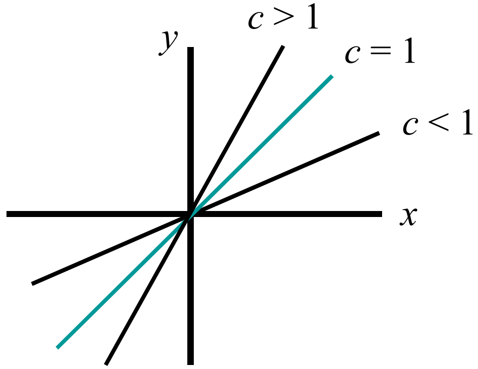
$y_2 = -\sqrt{25 - x^2}, -5 < x < 5$

Families of Solutions

- A solution to a 1st-order DE containing an arbitrary constant represents a set $G(x, y, c) = 0$ of solutions is called a one-parameter family of solutions.
- For n^{th}-order DE, an n-parameter family of solutions can be represented as

  $$G(x, y, c_1, c_2, \ldots, c_n) = 0.$$

  If the parameters $c_1, c_2, \ldots, c_n$ are resolved, then it's called a particular solution of the DE.
- Example:
  $y - cx = 0$ is a family of solutions of $xy' - y = 0$.



Singular Solutions

- **Definition:** A singular solution is a solution that cannot be obtained by specializing any of the parameters in the family of solutions.

- Example:
  Both $y = x^4/16$ and $y = 0$ are solutions of $dy/dx = xy^{1/2}$ on the interval $(-\infty, \infty)$. The ODE possesses the family of solutions $y = (x^2/4 + c)^2$. However, $y = 0$ is not in the family of solutions.

General Solutions

- **Definition:** If every solution of an n^{th}-order ODE $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ on an interval I can be obtained from an n-parameter family of equations $G(x, y, c_1, c_2, \ldots, c_n) = 0$ by appropriate choices of the parameters c_i, $i = 1, 2, \ldots, n$, we then say that the n-parameter family of equation is the general solution of the D.E.

Example: Two-Parameter Family

- The functions $x = c_1\cos 4t$ and $x = c_2\sin 4t$, where c_1 and c_2 are arbitrary constants, are solutions of $x'' + 16x = 0$.

  For $x = c_1\cos 4t$, the first two derivatives w.r.t. t are $x' = -4c_1 \sin 4t$ and $x'' = -16c_1\cos 4t$.

  Substituting x'' and x' into the DE gives

  $$x'' + 16x = -16c_1\cos 4t + 16(c_1\cos 4t) = 0.$$

  Similarly, for $x = c_2\sin 4t$, we have

  $$x'' + 16x = -16c_2\sin 4t + 16(c_2\sin 4t) = 0.$$

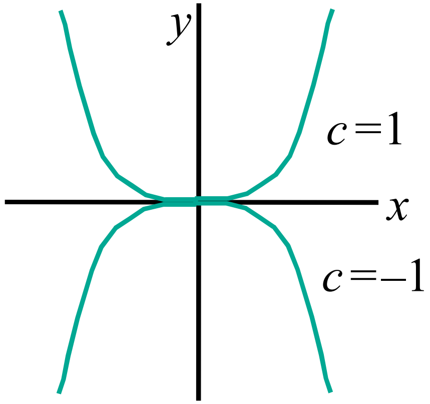
  Their linear combinations are a family of solutions.

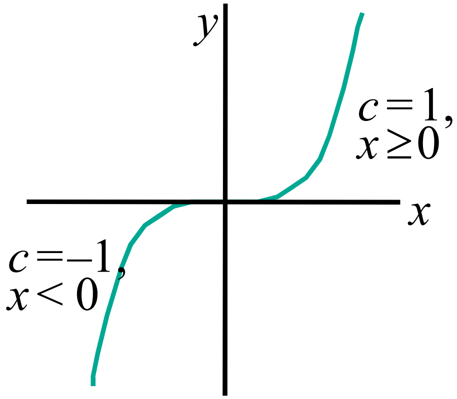
Example: Piecewise Solutions

- One can verify that $y = cx^4$ is a solution of $xy' - 4y = 0$ on the interval $(-\infty, \infty)$. The following piecewise defined solution is a particular solution of the ODE:

$$y = \begin{cases} -x^4, & x < 0 \\ x^4, & x \geq 0 \end{cases}$$

- This particular solution cannot be obtained by a single choice of c.





Initial Value Problem

- **Definition:**
  On some interval I containing x_0, the problem:

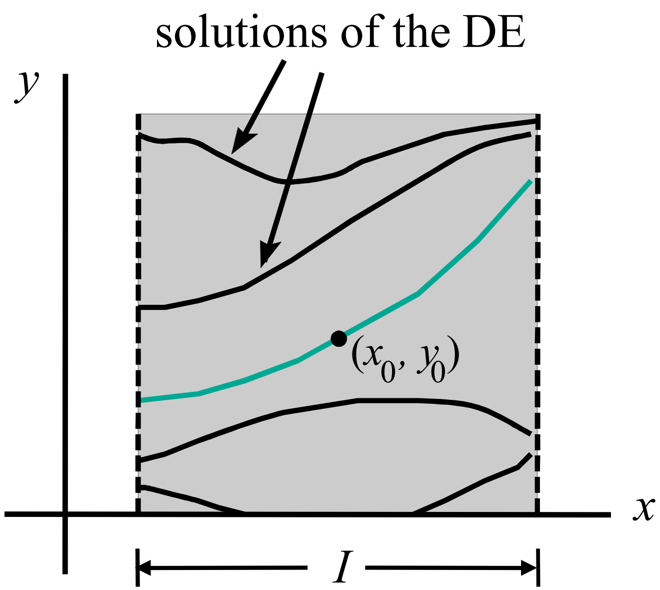
  Solve: $$\frac{d^n y}{dx^n} = f\left(x, y, y', \ldots, y^{(n-1)}\right)$$

  Subject to: $y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1},$

  where $y_0, y_1, \ldots, y_{n-1}$, are arbitrarily specified real constants, is called an initial value problem (IVP).
  The values of $y(x)$ and its first $n-1$ derivatives at x_0 are called initial conditions.

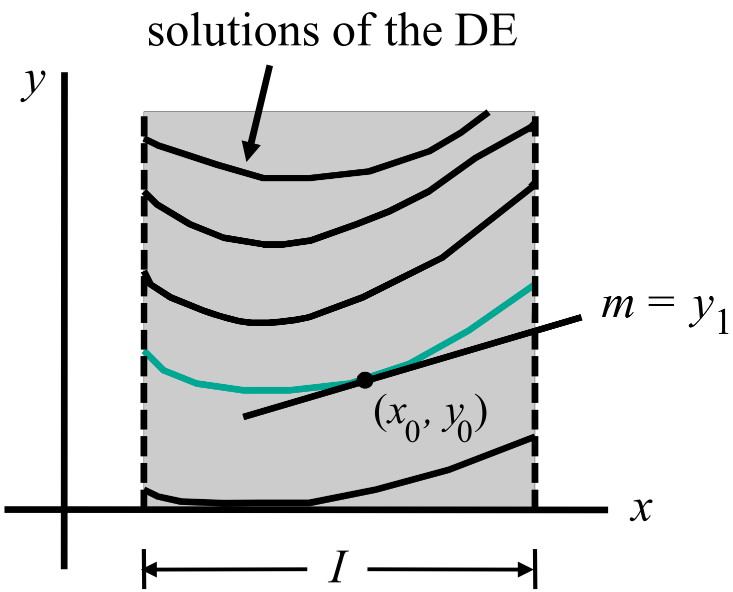
First Order IVP

- A first order IVP tries to solve $dy/dx = f(x, y)$, subject to $y(x_0) = y_0$. In geometric term, we are seeking a solution so that the solution curve passes through the prescribed point (x_0, y_0).



Second Order IVP

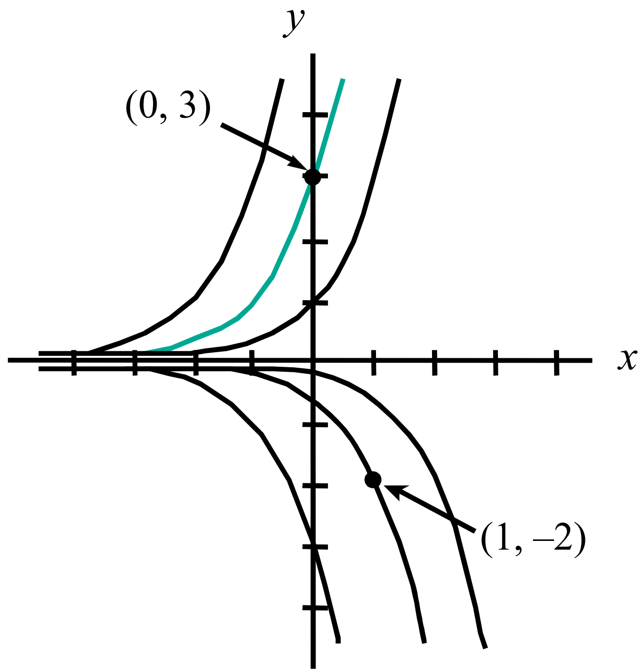
- A second order IVP tries to solve $d^2y/dx^2 = f(x, y, y')$, subject to $y(x_0) = y_0$, $y'(x_0) = y_1$. In geometric term, we are seeking a solution so that the solution curve not only passes through the prescribed point (x_0, y_0), but also with a slope y_1 at this point.



Example: 1st-Order IVPs

- It is easy to verify that $y = ce^x$ is a one-parameter family of solutions of the simple first-order equation $y' = y$ on the interval $(-\infty, \infty)$. If $y(0) = 3$, we have

$$3 = ce^0 = c$$



$\rightarrow y = 3e^x$ is a solution of IVP:
$y' = y,\ y(0) = 3.$

Existence of Unique Solution

- Two key questions of solving an IVP are:
  - Do solutions exist for the differential equation?
  - Given an initial condition, is the solution unique?

- Examples:
  - The IVP $y' = 1/x$, $y(0) = 0$ has no solution. By integration, we have $y(x) = \ln|x| + c$; but $\ln|x|$ is not defined at 0!
  - The IVP $dy/dx = xy^{1/2}$, $y(0) = 0$ has at least two solutions: $y = 0$ and $y = x^4/16$.

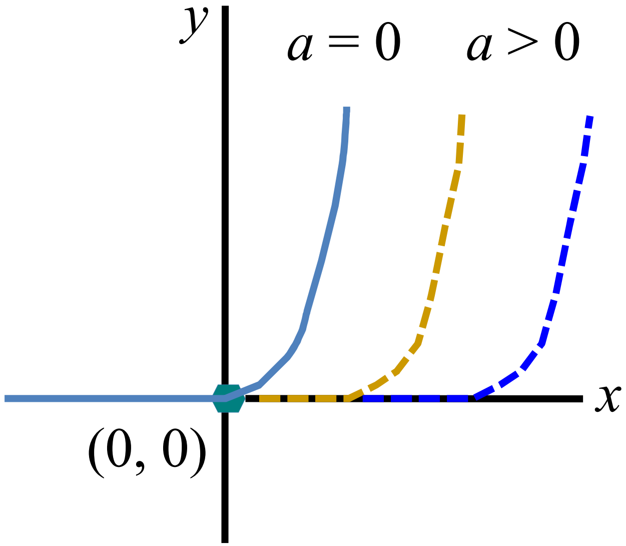
Example: Multiple IVP Solutions (1/2)

- Consider the IVP $dy/dx = xy^{1/2}$, $y(0) = 0$:
  The DE has a constant solution $y = 0$ and a family of solution

$$y = \left(\frac{x^2}{4} + c\right)^2.$$

The IVP has infinite solutions:
For any $a \geq 0$,

$$y = \begin{cases} 0, & x < a \\ \left(x^2 - a^2\right)^2 / 16, & x \geq a \end{cases}$$

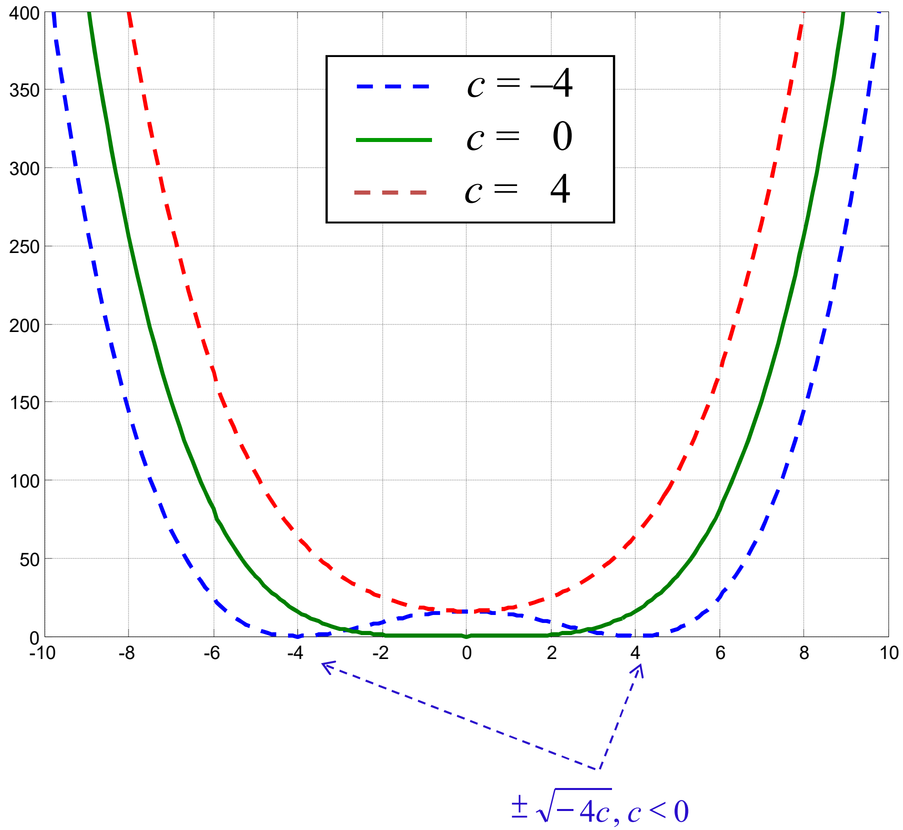


Example: Multiple IVP Solutions (2/2)

- Consider only the case $c \le 0$, let $c = -b,\ b \ge 0$:

$$y = \left(\frac{x^2}{4} + c\right)^2$$

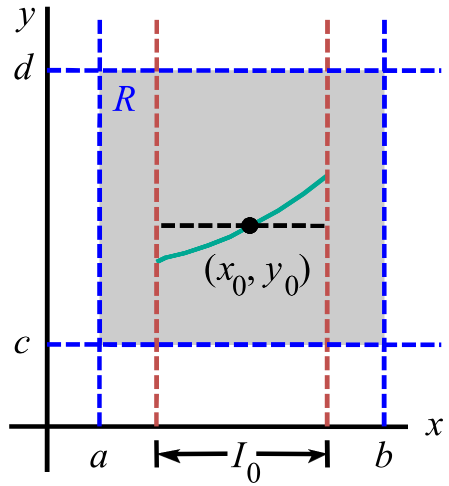
$$y = \left(\frac{x^2}{4} - \frac{4b}{4}\right)^2 = \left(\frac{x^2 - (2\sqrt{b})^2}{4}\right)^2 = (x^2 - a^2)^2/16, \quad a = 2\sqrt{b}$$



Existence and Uniqueness Theorem

- **Theorem:** Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le y \le d$, that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, there exist some interval I_0: $x_0 - h < x < x_0 + h$, $h > 0$, contained in $a \le x \le b$, and a unique function $y(x)$ defined on I_0 that is a solution of the first-order initial-value problem:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

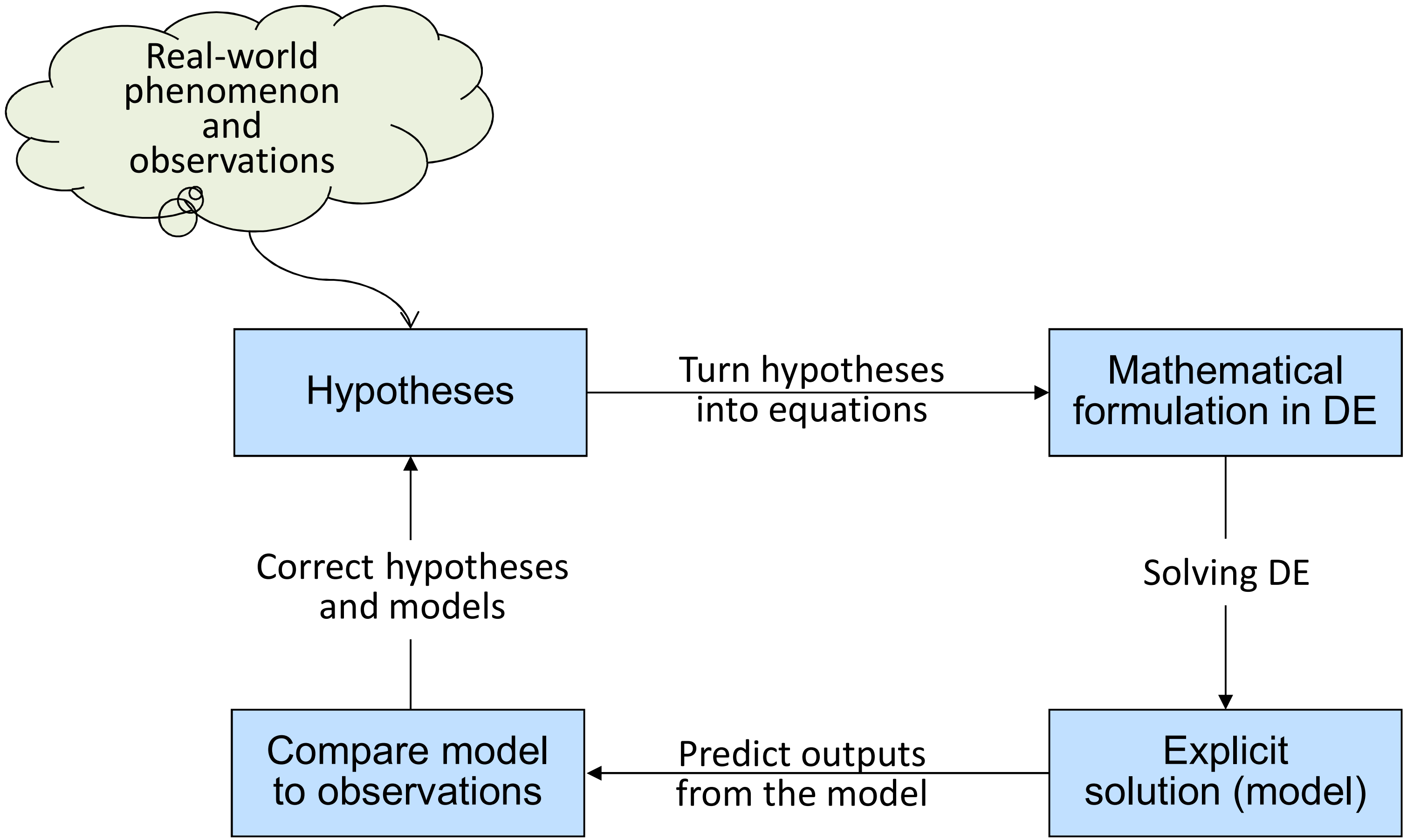


Example:

- Again, let's revisit the IVP: $dy/dx = xy^{1/2}$, $y(0)=0$.
  Since
  $$f(x, y) = xy^{1/2},$$
  and
  $$\partial f/\partial y = x/(2y^{1/2}),$$
  they are continuous in the upper half-plane defined by $y > 0$. Therefore, for any (x_0, y_0), $y_0 > 0$, there is an interval centered at x_0 on which the given DE has a unique solution.

  However, There is no unique solution for the IVP since $\partial f/\partial y$ is undefined at $(0, 0)$.

DE as Mathematical Models



Natural Growth and Decay Models

- The differential equation

$$\frac{dx}{dt} = kx, \quad \text{where } k \text{ is a constant.}$$

  is a widely used model for natural phenomena whose rate of change over time is proportional to its current population → what is the solution?

- If a population has birth and death rates β and δ, respectively. The differential change in size $P(t)$ of the population changes is

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{\beta P(t)\Delta t - \delta P(t)\Delta t}{\Delta t} = (\beta - \delta)P.$$

Falling Bodies

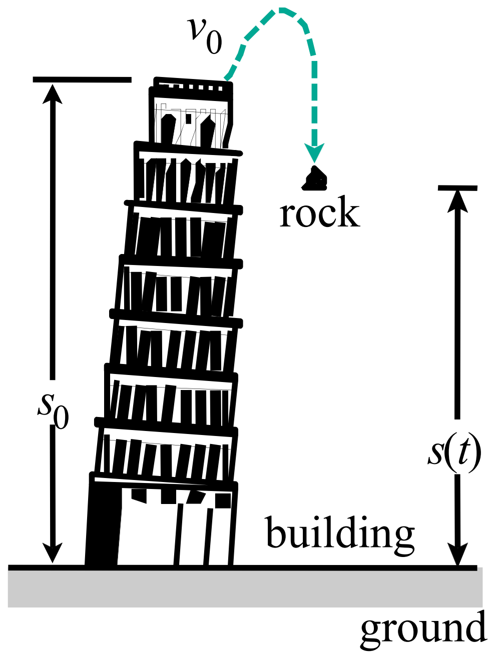
- Newton's second law of motion: $F = ma$
- Question: what is the position $s(t)$ of the rock relative to the ground at time t?

Acceleration of the rock: d^2s/dt^2

$$\rightarrow \quad m\frac{d^2s}{dt^2} = -mg \quad \rightarrow \quad \frac{d^2s}{dt^2} = -g$$

Model: $d^2s/dt^2 = -g$, $s(0) = s_0$, $s'(0) = v_0$.

Solution: $s(t) = -gt^2/2 + v_0t + s_0$



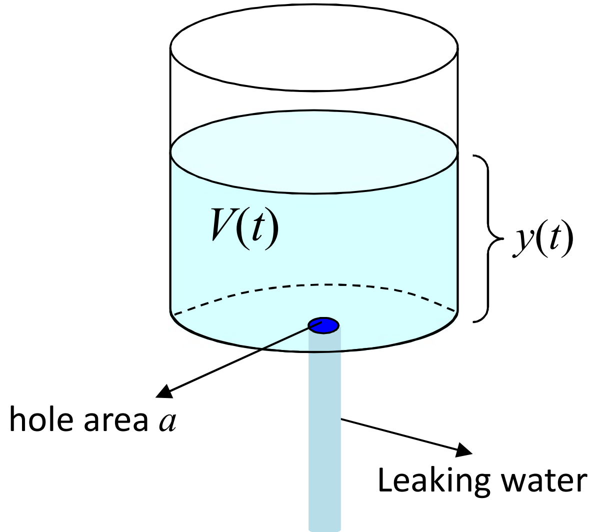
Torricelli's Model of a Draining Tank

- Torricelli's Law of draining tank:

$$\frac{dV}{dt} = -ac\sqrt{2gy}.$$

Derivation: Torricelli assumes that a drop of water from the surface escapes the hole at the speed

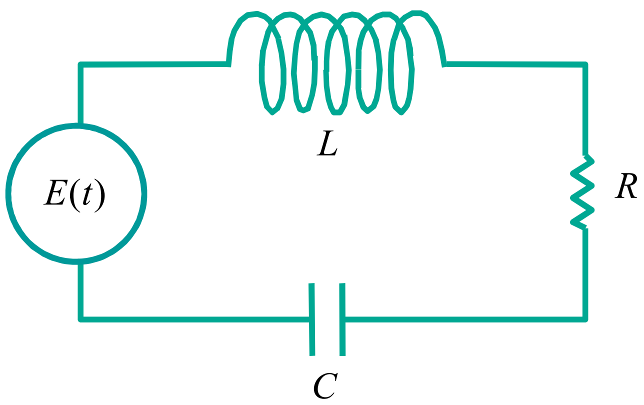
$$v = c\sqrt{2gy}.$$



Note: In Torricelli's law, $0 \le c \le 1$ is a constant parameter related to the viscosity of the liquid.

Series Circuit

- If $i(t) = dq/dt$ is the electric current across the circuit, the voltage drops across different electric components are:
  - Inductor: $v = L\dfrac{di}{dt}$
  - Resister: $v = Ri$
  - Capacitor: $v = \dfrac{1}{C}q$



- Kirchhoff's second law of circuits:

  Voltage drop = Impressed Voltage, that is:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$